

Algebraization of a Cartier divisor

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Abstract

We extend to pairs classical results of R. Elkik on lifting of homomorphisms and algebraization. In particular, we establish algebraization of an affine rig-smooth formal variety with a rig-smooth closed subvariety. This solves affirmatively a problem raised by M. Temkin and has applications to desingularization theory.

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Introduction

Motivation

In a fundamental work [1], Artin studied how various algebraic structures over the completion or henselization of a ring A along an ideal m can be approximated with a structure over A . Some his results were extended in a fundamental paper of Elkik [3]. In particular, Elkik studied approximation and algebraization of rig-smooth schemes and formal schemes defined over a complete or henselian ring. Note also that very recently several constructions of [3] were clarified and simplified by Gabber and Ramero in [4, Chapter 5].

The results of [3] have various important applications in algebraic geometry and related fields. They are especially useful when one wants to reduce questions about general schemes to the case of varieties. One of such applications is resolution of singularities of quasi-excellent schemes, see [8] and [9]. However, the reliance of the latter papers on [3] imposed some restrictions due to the fact that a basic datum involved in the strongest desingularization results (often referred to as embedded desingularization) is rather complicated, and its algebraization is not covered by [3]. This raised a natural quest for extending the results of [3] to more complicated algebraic structures. For example, in the end of [8, §1.1] Temkin noted that probably Elkik's algebraization of rig-smooth formal schemes can be extended to pairs consisting of a rig-smooth formal scheme and a rig-snc divisor. In this paper, we explore a slightly different direction. We consider only a rig-smooth formal subscheme but do not restrict its codimension. Our results will be used by Temkin to establish strong embedded desingularization of quasi-excellent schemes of characteristic zero.

The method, results, and relation to [3]

Our main goal is to prove Theorem 49, which is a natural generalization of [3, Theorem 7] to rig-smooth pairs. It should be noted that by loc.cit. we can algebraize the scheme and its subscheme independently, and it is also not difficult to obtain a morphism between the algebraizations. The main difficulty is to guarantee that the connecting homomorphism of rings is surjective (i.e. the morphism of schemes is a closed immersion). In order to prove this, we have to generalize almost all essential results of [3]. Here is the list of them:

- Theorem 26 shows the lifting property with values in completions for a pair of rig-smooth algebras. It generalizes Theorem 1.
- Theorem 28 shows that every homomorphism of a pair of rig-smooth algebras to a pair of completions can be approximated by a homomorphism to a pair of henselizations. It generalizes Theorem 2 bis.
- Theorem 32 shows the lifting property with values in henselizations for a pair of rig-smooth algebras. It generalizes Theorem 2 and is just an equivalent form of Theorem 28.

- Theorem 42 shows that every surjective homomorphism of modules being projective over the complement of $V(\mathfrak{a})$ is algebraizable. It generalizes Theorem 3.
- Theorem 25 shows the lifting property for a pair of modules. It generalizes Lemma on page 572.
- Theorem 51 is a corollary of Theorem 49. It shows that every rig-smooth Cartier divisor on a rig-smooth affine formal scheme is algebraizable.

We also added Proposition 38, whose analogue does not appear in [3] but can be found in [9, Prop. 3.3.1].

It should be noted that Gabber and Ramero introduce a different Jacobian ideal than the one used in [3]. Set-theoretically, it defines the same set of non-smooth points on a scheme but it is more flexible from the scheme-theoretic point of view. In our paper, we use the Gabber-Ramero ideal because it allows us to avoid computation at all and speak on the language of diagrams.

In the text, we have several restrictions. One of them is the number of generators for the defining ideal \mathfrak{a} . From the one hand, this restriction is essential only in Proposition 45, because the original result from [3] also has this restriction and we use it in our proof. From the other hand, for the sake of simplification, we intentionally write all other results in the paper in the case of the principal ideal. Using the same induction trick as used in [3], one can easily extend the results to the general case.

In our statements, we pay attention to different numerical parameters of homomorphisms liftings. One of the most important is the number h such that $\mathfrak{a}^h \subseteq \bar{H}_{B/A}$. This number is called a conductor in [9]. However, we usually take sums of maximums of conductors in situations when one can use just the sums of them. A careful reading of the proofs may give slightly more general statements than we proved but this is not our main aim.

General plan of the proof of Theorem 49

In a sense, we directly generalize the proof of [3, Theorem 7] with some technical simplifications. We start with a henselian pair (A, \mathfrak{a}) , where $\mathfrak{a} = (t) \subseteq A$ is a principal ideal. Then the problem is to algebraize a surjective homomorphism $B \rightarrow \bar{B}$ of formally finitely generated \hat{A} -algebras being formally smooth over the complement of $V(\mathfrak{a})$. We solve the problem in two steps. The first one is to show the result in a particular case, when we can compute everything explicitly. The second step is to reduce of the general case to this one. The reduction is done by algebraization of a pair of modules being projective over the complement of $V(\mathfrak{a})$. Now, let us describe the two steps with more details.

Step 1 If we want to algebraize $B = \hat{A}\{X\}/J$, where $J = (g_1, \dots, g_k, e)$, we can find some $g_1^0, \dots, g_k^0, e^0 \in A[X]$ such that

$$g_i \equiv g_i^0 \pmod{\mathfrak{a}^n} \quad \text{and} \quad e \equiv e^0 \pmod{\mathfrak{a}^n}.$$

If e is an idempotent modulo g_i and g_i form a basis of $(J/J^2)_t$ (this determines our special case), we automatically get that $B^0 = A[X]/(g_1^0, \dots, g_k^0, e^0)$ is smooth and, if n was sufficiently large, we automatically derive from this that B and \widehat{B}^0 are isomorphic. In order to find such g_i^0 and e^0 , we write down the equation saying that e is an idempotent modulo g_i and solve this equation using lifting Theorem 2 of [3].

If we want to algebraize a pair of algebras given by $J \subseteq \bar{J} \subseteq \widehat{A}\{X\}$ with $B = \widehat{A}\{X\}/J$ and $\bar{B} = \widehat{A}\{X\}/\bar{J}$, we should pose the condition above on both ideals J and \bar{J} and also add another restriction: we should have an isomorphism of the following form

$$(J/J^2) \otimes_B \bar{B}_t = (\bar{J}/\bar{J}^2)_t \oplus D$$

where D is \bar{B}_t -free. Then we can also write down an explicit system of equations on the generators of J and \bar{J} and find its solution due to Theorem 2 of [3].

Step 2 In order to reduce an arbitrary case to the particular one, we should replace the quotient homomorphism $B \rightarrow \bar{B}$ by homomorphisms of the form $S_B\{J/J^2\} \rightarrow S_{\bar{B}}\{\bar{J}/\bar{J}^2\}$ and $B\{X\} \rightarrow \bar{B}$. Using separated Kähler differentials, one can show that the reduction is always possible. In order to come back to the initial algebras, it is enough to algebraize a pair of modules $P \rightarrow \bar{P}$, where P_t and \bar{P}_t are projective B_t and \bar{B}_t -modules, respectively (one of the steps uses $P = J/J^2$ and $\bar{P} = \bar{J}/\bar{J}^2$). The case of modules is easier because we can replace, say P , by the beginning of its free resolution $B^m \xrightarrow{L} B^n \rightarrow P$. Then we manipulate with finite set of matrices like L and similar ones.

Structure of the paper

In the first section, we fix some terminology and notation. In Section 2, we combine ideas of [4] with [3] and provide scheme-theoretic basis for our needs. Section 3 is devoted to lifting theorems for modules and algebras. This is a technical core of the paper. Approximation of pairs of henselizations and pairs of complete algebras is considered in Section 4. In the final section, we prove our main result that is Theorem 49 and its corollary – Theorem 51.

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1 Conventions

Through out the text we assume that our rings are associative, commutative and with an identity element. We also suppose that all our rings are Noetherian. If we are given a ring A with an ideal \mathfrak{a} and an étale homomorphism $A \rightarrow B$. We will say that B is strictly étale if $A/\mathfrak{a} = B/\mathfrak{a}B$. The pair (A, \mathfrak{a}) is said to be henselian if, for every strictly étale homomorphism $\nu: A \rightarrow B$, there is a homomorphism $\mu: B \rightarrow A$ such that $\mu \circ \nu = Id_A$.

If A is an arbitrary ring and $\mathfrak{a} \subseteq A$ is an arbitrary ideal, we define the following ring

$$A^h = \varinjlim \{B \mid A \rightarrow B \text{ is strictly étale}\}$$

The ring A^h is called the henselization of A with respect to \mathfrak{a} . The completion of a ring A with \mathfrak{a} -adic topology will be denoted by \hat{A} .

For any finite set $x = \{x_1, \dots, x_n\}$, the rings $A[x]$, $A[x]^h$, and $A\{x\}$ denote the ring of polynomials, its henselization with respect to $\mathfrak{a}[x]$, and the completion in $\mathfrak{a}[x]$ -adic topology, respectively. Every derivation $\partial_i = \partial/\partial x_i$ of the ring $A[x]$ uniquely extends to derivations in $A[x]^h$ and $A\{x\}$ and its extensions will be denoted by the same name ∂_i . If we are given elements $f_1, \dots, f_n \in R$, where R is either $A[x]$, $A[x]^h$, or $A\{x\}$, then $\Delta^m(f_1, \dots, f_n)$ will be the ideal generated by all m -minors of the matrix $(\partial_j f_i)$.

For any ring A the category of A -modules will be denoted by $A\text{-mod}$ and the category of finite A -modules by $A\text{-mod}^0$.

Let again A be a ring and $\mathfrak{a} \subseteq A$ is an arbitrary ideal. We supply every A -module with \mathfrak{a} -adic topology. In this case, for every A -module M , every derivation $d: A \rightarrow M$ is continuous. Now, assume that A is \mathfrak{a} -adically complete. We will say that an A -algebra B is formally finitely generated if there is a dense finitely generated A -algebra. In other words, B is a quotient of $A\{x\}$ for some finite set x . Let us note, that every formally finitely generated A -algebra is complete and separated by definition. Moreover, every finitely generated module over B is complete and separated.

Suppose that a formally finitely generated algebra B is presented as follows

$$B = A\{x_1, \dots, x_n\}/J, \quad \text{where} \quad J = (f_1, \dots, f_q)$$

and for some natural p , we are given $(\alpha) = (\alpha_1, \dots, \alpha_p)$, where

$$1 \leq \alpha_1 < \dots < \alpha_p \leq q$$

we define the following ideals of the ring $A\{x_1, \dots, x_n\}$: the ideal $\Delta_{(\alpha)}$ is the ideal generated by the q -minors of the matrix $(\partial f_{\alpha_i}/\partial x_j)_{\alpha_i \in (\alpha), j \in [1, n]}$, the ideal J_α is the ideal generated by f_{α_i} with $\alpha_i \in \alpha$. Then we define $H_J \subseteq B$ as the image of the ideal $\sum_{\alpha} \Delta_{(\alpha)}(J_\alpha : J)$.

2 Scheme-Theoretic support

2.1 The support of singularities

2.1.1 Smoothness

Let A be a ring, B is an A -algebra, and N is a B -module. Recall that an extension $E = (D, \varepsilon, i)$ of B by N is the following exact sequence

$$E : 0 \rightarrow N \xrightarrow{i} D \xrightarrow{\varepsilon} B \rightarrow 0$$

where D is an A -algebra, ε is a homomorphism of A -algebras, $i(N) = \ker \varepsilon$ is a square zero ideal such that the action of B on $N = N/N^2$ coincides with the initial action on N . The two such extensions $E = (D, \varepsilon, i)$ and $E' = (D', \varepsilon', i')$ are said to be isomorphic if there is an A -isomorphism $\psi : D \rightarrow D'$ inducing the identity maps on N and D . Morphisms between two extensions are defined in the obvious way.

Recall that, for any extension $E = (D, \varepsilon, i)$, any A -algebras homomorphism $\phi : B' \rightarrow B$, and any B -module homomorphism $\xi : N \rightarrow M$, there are unique extensions $E\phi$ and ξE such that the following diagrams are commutative

$$\begin{array}{ccccccccc} E\phi : & 0 & \longrightarrow & N & \longrightarrow & D' & \longrightarrow & B' & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ E : & 0 & \longrightarrow & N & \xrightarrow{i} & D & \xrightarrow{\varepsilon} & B & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} E : & 0 & \longrightarrow & N & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\ & & & \downarrow \xi & & \downarrow & & \parallel & & \\ \xi E : & 0 & \longrightarrow & M & \longrightarrow & D'' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The set of all extensions of B by N will be denoted by $E_A(B, N)$. It is an abelian group with respect to the Baer sum and the multiplication by the elements of B gives a structure of a B -module. Now, we can define the following ideal

$$H_{B/A} = \bigcap_{N \in B\text{-mod}} \text{ann}_B(E_A(B, N))$$

Note that, if we say that B is smooth A -algebra, we do not necessarily assume that B is finitely generated. We do not need this finiteness condition to describe the properties of the ideal $H_{B/A}$. However, in all applications this condition will be assumed and explicitly stated.

In [3, Section 0.2], another ideal is used instead of $H_{B/A}$. The ideal $H_{B/A}$ was introduced in [4, Chapter 5, Section 5.4] in a slightly different way. Due to Gabber's ideal, we can significantly simplify computational part of methods used in [3].

Lemma 1. *Let A be a ring, R be a smooth A -algebra, J an ideal of R , and B is an A -algebra such that the following sequence is exact*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then the natural map

$$\oplus_{\xi \in \text{Hom}_B(J/J^2, N)} E_A(B, \xi): \bigoplus_{\xi \in \text{Hom}_B(J/J^2, N)} E_A(B, J/J^2) \rightarrow E_A(B, N)$$

is surjective.

Proof. Consider the following commutative diagram

$$E: \begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & D & \longrightarrow & B \longrightarrow 0 \end{array}$$

where the exact sequence E is an element of $E_A(B, N)$ and we are going to produce the dotted lines. Since R is A -smooth, there is an A -lifting $\phi: R \rightarrow D$ of the identity map Id_B . Then ξ is the restriction of ϕ on J . Since $N^2 = 0$, $\phi(J^2) = 0$. Therefore, we have the following commutative diagram

$$E = \bar{\xi}T: \begin{array}{ccccccc} 0 & \longrightarrow & J/J^2 & \longrightarrow & R/J^2 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \bar{\xi} & & \downarrow \bar{\phi} & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & D & \longrightarrow & B \longrightarrow 0 \end{array}$$

where $\bar{\phi}$ and $\bar{\xi}$ are the maps induced by ϕ and ξ , respectively. Hence, the element E belongs to the image of $E_A(B, \bar{\xi})$. \square

Corollary 2. *Let A be a ring, R a smooth A -algebra, J an ideal of R , and B is an A -algebra such that the following sequence is exact*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then

$$H_{B/A} = \text{ann}_B(E_A(B, J/J^2)).$$

Lemma 3. *Let A be a ring, R a smooth A -algebra, J an ideal of R , and B is an A -algebra such that the following sequence is exact*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then we have

$$H_{B/A} = \text{ann}_B(\text{Hom}_B(J/J^2, J/J^2)/\text{Der}_A(R, J/J^2))$$

Proof. By Corollary 2, we have

$$H_{B/A} = \text{ann}_B (E_A(B, J/J^2)).$$

Let $x \in B$ be an arbitrary element, we can write the following commutative diagram

$$\begin{array}{ccccccc} T : & 0 & \longrightarrow & J/J^2 & \longrightarrow & R/J^2 & \longrightarrow & B & \longrightarrow & 0 \\ & & & \downarrow x & & \downarrow & & \parallel & & \\ xT : & 0 & \longrightarrow & J/J^2 & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The element x belongs $H_{B/A}$ if and only if $xT = 0$. The latter means xT splits. In other words, the multiplication by x extends to an A -derivation

$$d: R/J^2 \rightarrow J/J^2.$$

Since J/J^2 is a module over $R/J = B$,

$$\text{Der}_A(R/J^2, J/J^2) = \text{Der}_A(R, J/J^2).$$

□

The ideal $H_{B/A}$ describes A -smoothness in the following sense.

Lemma 4. *Let A be a ring, and B a finitely generated A -algebra. Then, for every prime ideal $\mathfrak{p} \subseteq B$, $B_{\mathfrak{p}}$ is A -smooth if and only if $H_{B/A} \not\subseteq \mathfrak{p}$.*

Proof. See Lemma 5.4.2 (ii) of [4, Chapter 5, Section 5.4].

□

2.1.2 Separated differentials

Let A be a ring and $\mathfrak{a} \subseteq A$ is an arbitrary ideal such that A is \mathfrak{a} -adically complete. Let B is a formally finitely generated A -algebra. We denote by $\Omega_{B/A}^s$ the Hausdorff quotient of $\Omega_{B/A}$. Then $\Omega_{B/A}^s$ becomes a finite B -module. In particular, $\Omega_{B/A}^s = \widehat{\Omega}_{B/A}$. If $B = A\{x\}$ is a ring of convergent series, the module $\Omega_{A\{x\}/A}^s$ is a free module with generators dx . In our particular case, (20.7.17) and (20.7.20) of [5] gives the following results.

Proposition 5 (The first fundamental sequence). *Let A be a topological ring, $B \rightarrow C$ is a homomorphism of formally finitely generated A -algebras, then*

1. *the following sequence is exact*

$$\Omega_{B/A}^s \otimes_B C \rightarrow \Omega_{C/A}^s \rightarrow \Omega_{C/B}^s \rightarrow 0$$

2. *the sequence is split exact if and only if, for every finite B -module N , any derivation from B to N over A extends to a derivation from C to N .*

Proposition 6 (The second fundamental sequence). *Let A be a topological ring, $B \rightarrow C$ is a topologically surjective homomorphism of topological A -algebras with the kernel J . Then*

1. *the following sequence is exact*

$$J/J^2 \rightarrow \Omega_{B/A}^s \otimes_B C \rightarrow \Omega_{C/A}^s \rightarrow 0$$

2. *the sequence is split exact if and only if the following sequence splits*

$$0 \rightarrow J/J^2 \rightarrow B/J^2 \rightarrow C \rightarrow 0$$

2.1.3 Formal smoothness

Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal such that A is \mathfrak{a} -adically complete. Let B be a formally finitely generated A -algebra, then we define

$$\bar{H}_{B/A} = \bigcap_{M \in B\text{-mod}^0} \text{ann}_B(\mathbb{E}_A(B, M))$$

We will say that algebra B is formally smooth over the complement of $V(\mathfrak{a})$ if $\mathfrak{a} \subseteq r(H_{B/A})$.

Lemma 7. *Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal such that A is \mathfrak{a} -adically complete, B is formally finitely generated A -algebra, and N is a finite B -module. Assume that R is the completion of a smooth finitely generated A -algebra R_0 such that*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then the natural map

$$\bigoplus_{\xi \in \text{Hom}_B(J/J^2, N)} \mathbb{E}_A(B, J/J^2) \xrightarrow{\oplus \mathbb{E}_A(B, \xi)} \mathbb{E}_A(B, N)$$

is surjective.

Proof. Let N be an arbitrary finitely generated B -module and

$$0 \rightarrow N \rightarrow D \rightarrow B \rightarrow 0$$

is an element of $\mathbb{E}_A(B, N)$. Since B and N are Noetherian, the ring D is also Noetherian. Moreover, B is \mathfrak{a} -adically complete by definition and N is finitely generated B -module, thus, N is also \mathfrak{a} -adically complete. Since D is Noetherian \mathfrak{a} -adic topology from D induces \mathfrak{a} -adic topology on N . Since completion is an exact functor, we see that D is complete.

Now, consider the following diagram

$$\begin{array}{ccccc} R_0 & \longrightarrow & B & & \\ \downarrow \phi & \searrow & \parallel & & \\ D & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where ϕ is a lifting of the homomorphism from R_0 to B . It exists because of A -smoothness of R_0 . But D is complete, therefore, there is a unique extension $\hat{\phi}$ of ϕ to R . Moreover, since $\hat{\phi}$ is a lifting, it maps J^2 to zero. Now, we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J/J^2 & \longrightarrow & R/J^2 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & D & \longrightarrow & B \longrightarrow 0 \end{array}$$

where φ is the homomorphism induced by $\hat{\phi}$, and ξ is the restriction of φ to J/J^2 . The rest of the proof is a word by word repetition of the end of the proof of Lemma 1. \square

Corollary 8. *Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal such that A is \mathfrak{a} -adically complete, and B is a formally finitely generated A -algebra. Assume that R is the completion of a smooth finitely generated A -algebra R_0 such that*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then

$$\bar{H}_{B/A} = \text{ann}_B(\text{E}_A(B, J/J^2)).$$

Lemma 9. *Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal such that A is \mathfrak{a} -adically complete, and B is a formally finitely generated A -algebra. Assume that R is the completion of a smooth finitely generated A -algebra R_0 such that*

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

Then

$$\bar{H}_{B/A} = \text{ann}_B(\text{Hom}_B(J/J^2, J/J^2)/\text{Der}_A(B, J/J^2))$$

Proof. By Corollary 8, we have

$$\bar{H}_{B/A} = \text{ann}_B(\text{E}_A(B, J/J^2)).$$

Let $x \in B$ be an arbitrary element, we can write the following commutative diagram

$$\begin{array}{ccccccc} T : & 0 & \longrightarrow & J/J^2 & \longrightarrow & R/J^2 & \longrightarrow B \longrightarrow 0 \\ & & & \downarrow x & & \downarrow & \parallel \\ xT : & 0 & \longrightarrow & J/J^2 & \longrightarrow & D & \longrightarrow B \longrightarrow 0 \end{array}$$

The element x belongs to $\bar{H}_{B/A}$ if and only if $xT = 0$. The latter means xT splits. In other words, the multiplication by x extends to an A -derivation

$$d: R/J^2 \rightarrow J/J^2.$$

Since J/J^2 is a module over $R/J = B$,

$$\mathrm{Der}_A(R/J^2, J/J^2) = \mathrm{Der}_A(R, J/J^2).$$

□

Lemma 10. *Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal, B is a finitely generated A -algebra. Then*

$$H_{B/A} \hat{B} = H_{B/A} \otimes_B \hat{B} = \bar{H}_{\hat{B}/\hat{A}}$$

Proof. Consider an arbitrary exact sequence

$$0 \rightarrow J \rightarrow A[x] \rightarrow B \rightarrow 0$$

where $x = \{x_1, \dots, x_n\}$ is a finite set of indeterminates. Then, by Lemma 3,

$$H_{B/A} = \mathrm{ann}_B (\mathrm{Hom}_B(J/J^2, J/J^2) / \mathrm{Hom}_B(\Omega_{A[x]/A} \otimes_{A[x]} B, J/J^2))$$

Since J/J^2 , $\Omega_{A[x]/A} \otimes_{A[x]} B$ are finitely generated B -modules and \hat{B} is a flat B -module, then

$$\begin{aligned} H_{B/A} \hat{B} &= H_{B/A} \otimes_B \hat{B} = \\ &= \mathrm{ann}_{\hat{B}} \left(\mathrm{Hom}_{\hat{B}}(\hat{J}/\hat{J}^2, \hat{J}/\hat{J}^2) / \mathrm{Hom}_{\hat{B}}(\Omega_{\hat{A}\{x\}/\hat{A}}^s \otimes_{\hat{A}\{x\}} \hat{B}, \hat{J}/\hat{J}^2) \right) = \\ &= \mathrm{ann}_{\hat{B}} \left(\mathrm{Hom}_{\hat{B}}(\hat{J}/\hat{J}^2, \hat{J}/\hat{J}^2) / \mathrm{Der}_{\hat{A}}(\hat{B}, \hat{J}/\hat{J}^2) \right) = \\ &= \bar{H}_{\hat{B}/\hat{A}} \end{aligned}$$

The latter equality holds because of Lemma 9. □

Remark 11. Lemma 10 shows that if an \hat{A} -algebra \bar{B} is a completion of a finitely generated A -algebra being smooth over the complement of $V(\mathfrak{a})$, then \bar{B} is formally smooth over the complement of $V(\hat{\mathfrak{a}})$. So, the condition of formal smoothness is necessary for algebraization of an \hat{A} -algebra \bar{B} by a smooth A -algebra.

Lemma 12. *Let A be a ring $\mathfrak{a} \subseteq A$ is an ideal such that A is \mathfrak{a} -adically complete, and B is a formally finitely generated A -algebra presented in the following form*

$$B = A\{x_1, \dots, x_n\}/J, \quad \text{where } J = (f_1, \dots, f_q)$$

Then

$$H_J \subseteq \bar{H}_{B/A} \quad \text{and} \quad r(H_J) = r(\bar{H}_{B/A})$$

Proof. The proof of the inclusion $H_J \subseteq \bar{H}_{B/A}$ is a word by word repetition of the proof of Lemma 5.4.6 of [4], where we should use Ω^s instead of Ω .

To proof coincidence of the radicals, it is enough to show that, if a prime ideal $\mathfrak{p} \subseteq B$ does not contain $\bar{H}_{B/A}$, then it does not contain H_J . Let $t \in \bar{H}_{B/A} \setminus \mathfrak{p}$.

Then the map $J/J^2 \rightarrow J/J^2$ given by $x \mapsto tx$ lifts to a derivation $A\{x\} \rightarrow J/J^2$ by definition. In particular, the sequence

$$0 \rightarrow J_{\mathfrak{p}}/J_{\mathfrak{p}}^2 \rightarrow \Omega_{A\{x\}/A}^s \otimes_{A\{x\}} B_{\mathfrak{p}} \rightarrow (\Omega_{B/A}^s)_{\mathfrak{p}} \rightarrow 0$$

is split exact. Since $B_{\mathfrak{p}}$ is local and f_1, \dots, f_q generate J , after reordering f_i , we may assume that, for some k , images of f_1, \dots, f_k form a basis of $J_{\mathfrak{p}}/J_{\mathfrak{p}}^2$. Since $B_{\mathfrak{p}}$ does not contain idempotents except 0 and 1, elements f_1, \dots, f_k generate $J_{\mathfrak{p}}$. In particular, there exists an element $t \in B \setminus \mathfrak{p}$ such that $tJ \subseteq (f_1, \dots, f_k)$.

Since $B_{\mathfrak{p}}$ is local, after reordering of x_i , we may assume that

$$df_1, \dots, df_k, dx_{k+1}, \dots, dx_n$$

is a basis of

$$\Omega_{A\{x\}/A}^s \otimes_{A\{x\}} B_{\mathfrak{p}}$$

because dx_i are generators, and the set df_1, \dots, df_k is a basis of a direct summand. Hence the element $h = \det(\partial f_i / \partial x_j)_{i,j \leq k}$ is invertible in $B_{\mathfrak{p}}$, thus, is not in \mathfrak{p} . Therefore, the element ht is not in \mathfrak{p} and belongs to H_J . \square

Remark 13. In particular, the notion of formal smoothness used in [3, Section III.4] coincides with the one used in this paper. Hence we can formally cite results from [3].

2.2 The support of non-projectivity

In this section, we define an analogue of the Gabber's ideal for modules. This ideal is used instead of the Fitting ideal to algebraize finite modules. It describes points where our module is not projective.

Let A be a ring and P is an arbitrary A -module. Then we define the ideal H_P by the formula

$$H_P = \bigcap_{M \in A\text{-mod}} \text{ann}_A(\text{Ext}_A^1(P, M))$$

Lemma 14. *Let A be a ring and we are given an exact sequence of A -modules*

$$0 \rightarrow K \rightarrow F_0 \rightarrow P \rightarrow 0$$

where F_0 is projective. Then the map

$$\oplus_g \text{Ext}_A^1(P, g): \bigoplus_{g \in \text{Hom}_A(K, M)} \text{Ext}_A^1(P, K) \rightarrow \text{Ext}_A^1(P, M)$$

is surjective.

Proof. Consider the following projective resolution for P

$$\begin{array}{ccccccccc} F_2 & \xrightarrow{d_2} & F_1 & \longrightarrow & F_0 & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow \xi & & \parallel & & \parallel & & \\ 0 & \longrightarrow & K & \longrightarrow & F_0 & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

where K is the cokernel of d_2 and ξ is the corresponding quotient homomorphism. The map ξ corresponds to an element $\bar{\xi}$ in $\text{Ext}_A^1(P, K)$. Let $\varphi: F_1 \rightarrow M$ be a homomorphism corresponding to some element $\bar{\varphi}$ of $\text{Ext}_A^1(P, M)$, then, by definition of ξ , there is a homomorphism $g: K \rightarrow M$ such that $g \circ \xi = \varphi$. Thus, the element $\bar{\varphi}$ belongs to the image of $\text{Ext}_A^1(P, g)$. \square

Corollary 15. *Let A be a ring and we are given an exact sequence of A -modules*

$$0 \rightarrow K \rightarrow F_0 \rightarrow P \rightarrow 0,$$

where F_0 is projective. Then

$$H_P = \text{ann}_A(\text{Ext}_A^1(P, K)).$$

Corollary 16. *Let A be a ring and P a finite A -module, then*

$$H_P = \bigcap_{M \in A\text{-mod}^0} \text{ann}_A(\text{Ext}_A^1(P, M)).$$

The following proposition explains the geometric meaning of the ideal H_P .

Proposition 17. *Let A be a ring, P a finite A -module, and $\mathfrak{p} \subseteq A$ is a prime ideal. Then $H_P \not\subseteq \mathfrak{p}$ if and only if $P_{\mathfrak{p}}$ is projective $A_{\mathfrak{p}}$ -module.*

Proof. Suppose that $H_P \not\subseteq \mathfrak{p}$ and let $t \in H_P \setminus \mathfrak{p}$. Consider some exact sequence $0 \rightarrow K \rightarrow F \rightarrow P \rightarrow 0$, where F is a finite free A -module. By the definition of H_P , the homomorphism $t: K \rightarrow K$ lifts to a homomorphism $F \rightarrow K$. After the localization by \mathfrak{p} , the multiplication by t becomes an isomorphism. So, the sequence $0 \rightarrow K_{\mathfrak{p}} \rightarrow F_{\mathfrak{p}} \rightarrow P_{\mathfrak{p}} \rightarrow 0$ splits.

Conversely, suppose $P_{\mathfrak{p}}$ is projective. By Corollary 15, it is enough to show that there is an $s \notin \mathfrak{p}$ such that $s: K \rightarrow K$ lifts to a homomorphism $F \rightarrow K$. Since the sequence $0 \rightarrow K_{\mathfrak{p}} \rightarrow F_{\mathfrak{p}} \rightarrow P_{\mathfrak{p}} \rightarrow 0$ splits, there is a splitting map $\psi: F_{\mathfrak{p}} \rightarrow K_{\mathfrak{p}}$. The module F is finite, hence, there is an element $s \notin \mathfrak{p}$ such that $s\psi: F \rightarrow K$ is a well-defined lifting of $s: K \rightarrow K$. \square

Proposition 18. *Let A be a ring, $\mathfrak{a} \subseteq A$ is an ideal such that A is a \mathfrak{a} -adically complete ring, and B is a formally finitely generated A -algebra. Then $\bar{H}_{B/A} \subseteq H_{\Omega_{B/A}^s}$.*

Proof. We represent our algebra B as the following quotient

$$0 \rightarrow J \rightarrow A\{x\} \rightarrow B \rightarrow 0$$

where $x = \{x_1, \dots, x_n\}$ is a finite set of indeterminates. By the second fundamental sequence, we have

$$0 \longrightarrow K \longrightarrow J/J^2 \longrightarrow \Omega_{A\{x\}/A}^s \otimes_{A\{x\}} B \longrightarrow \Omega_{B/A}^s \longrightarrow 0$$

Then by Lemmas 3 and 9, we have

$$\begin{aligned} E_A(B, J/J^2) &= \text{Hom}_B(J/J^2, J/J^2) / \text{Hom}_B(\Omega_{A\{x\}/A}^s \otimes_{A\{x\}} B, J/J^2) \\ \text{Ext}_B^1(\Omega_{B/A}^s, J/J^2) &= \text{Hom}_B((J/J^2)/K, J/J^2) / \text{Hom}_B(\Omega_{A\{x\}/A}^s \otimes_{A\{x\}} B, J/J^2) \end{aligned}$$

Hence, the inclusion

$$\text{Ext}_B^1(\Omega_{B/A}^s, J/J^2) \subseteq E_A(B, J/J^2)$$

holds. Therefore, we have

$$\bar{H}_{B/A} = \text{ann}_B(E_A(B, J/J^2)) \subseteq \text{ann}_B(\text{Ext}_B^1(\Omega_{B/A}^s, J/J^2)) = H_{\Omega_{B/A}^s}$$

□

Remark 19. Similarly to the proof of Proposition 18, one can show the following statement. If A is a ring and B is a finitely generated A -algebra, then

$$H_{B/A} \subseteq H_{\Omega_{B/A}}$$

But we will not need this fact.

3 Lifting of pairs of homomorphisms

3.1 Lifting of module homomorphisms

Lemma 20. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, and we are given an exact sequence of A -modules*

$$0 \rightarrow K \rightarrow M \xrightarrow{\varphi} N \rightarrow 0$$

Let φ_n be the restriction of φ on $\mathfrak{a}^n M$ and φ^d is the induced map from $M/\mathfrak{a}^d M \rightarrow N/\mathfrak{a}^d N$. Suppose that, for some natural number c and for all $n \geq c$, we have the following family of exact sequences

$$0 \rightarrow \mathfrak{a}^{n-c} P_n \rightarrow \mathfrak{a}^n M \xrightarrow{\varphi_n} \mathfrak{a}^n N \rightarrow 0,$$

where $P_n \subseteq \ker \varphi$. Then, for every pair (n, d) with $n \leq d$, we also have the following family of exact sequences

$$0 \rightarrow \mathfrak{a}^{n-c} P_n^d \rightarrow \mathfrak{a}^n M / \mathfrak{a}^d M \xrightarrow{\varphi_n^d} \mathfrak{a}^n N / \mathfrak{a}^d N \rightarrow 0,$$

where φ_n^d is the induced by φ map and $P_n^d \subseteq \ker \varphi_n^d$.

Moreover, if for any two pairs (n, d) and (n', d') such that $n \geq n'$ and $d \geq d'$, $P_n \subseteq P_{n'}$, the corresponding quotient map induces a well-defined map $P_n^d \rightarrow P_{n'}^{d'}$.

Proof. We define P_n^d as the image of P_n under the corresponding quotient map. Now, consider an element p of the kernel of φ_n^d . By definition, $p = t^n a$, where $a \in M$. Then $\varphi^d(t^n a) = 0$ in $N / \mathfrak{a}^d N$. Hence

$$\varphi(t^n a) = t^d h = t^d \varphi(b).$$

Therefore, $\varphi(t^n a - t^d b) = 0$. By the hypothesis, the element $k = t^n(a - t^{d-n}b)$ can be presented as $t^{n-c}r$, where $r \in P_n$. So, we have $p = t^{n-c}r$ in $M / \mathfrak{a}^d M$. \square

Lemma 21. Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, and we are given the following exact sequence of A -modules

$$0 \rightarrow R \rightarrow M \xrightarrow{\varphi} N \rightarrow 0$$

Let r and c be integral numbers such that

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$;
2. $\mathfrak{a}^n M \cap R = \mathfrak{a}^{n-c}(\mathfrak{a}^c M \cap R)$ for all $n \geq c$.

Let k , n , and h be a triple of integral numbers satisfying the conditions

$$k > n, \quad n > c + h, \quad n > r + h.$$

On the following diagram, $\varphi_{n,k}$ is the map induced by φ and $R_{n,k}$ is its kernel

$$0 \rightarrow R_{n,k} \rightarrow \mathfrak{a}^n M / \mathfrak{a}^k M \xrightarrow{\varphi_{n,k}} \mathfrak{a}^n N / \mathfrak{a}^k N \rightarrow 0$$

Suppose also that the following diagram with solid arrows only is given

$$\begin{array}{ccccc} R_{n,k} & \xrightarrow{\nu} & R_{n-h,k} & \xrightarrow{\mu} & R_{n-h,k-h} \\ \uparrow g & & & \nearrow t^h g' & \\ A^d & \xrightarrow{Q} & A^p & & \end{array}$$

where ν and μ are the natural maps, and we have $gQ = 0$. Then there exists a homomorphism

$$g' : A^p \rightarrow R_{n-h,k-h}$$

such that $g'Q = 0$ and $\mu \circ \nu \circ g = t^h g'$.

Proof. Firstly, by Lemma 20, $R_{n,k} = \mathfrak{a}^{n-c} \tilde{R}_{n,k}$, where $\tilde{R}_{n,k} \subseteq \ker \varphi_{n,k}$. Thus, we have the following diagram

$$\begin{array}{ccccc} & & \mathfrak{a}^{n-c} \tilde{R}_{n,k} & \xrightarrow{\nu} & \mathfrak{a}^{n-h-c} \tilde{R}_{n-h,k} & \xrightarrow{\mu} & \mathfrak{a}^{n-h-c} \tilde{R}_{n-h,k-h} \\ & & \uparrow g & & & & \\ A^d & \xrightarrow{Q} & A^p & & & & \end{array}$$

Hence, we have an equality $\nu \circ g = t^h \bar{f}$ for some $\bar{f}: A^p \rightarrow \mathfrak{a}^{n-h-c} \tilde{R}_{n-h,k}$. Its lifting $A^p \rightarrow \mathfrak{a}^{n-h} M$ will be denoted by f .

We will show that $\mu \circ \bar{f} \circ Q = 0$. Let q_1, \dots, q_d be the generators of the image of Q . Then, we know that

$$t^h \bar{f}(q_i) = 0 \text{ in } \mathfrak{a}^{n-h} M / \mathfrak{a}^k M.$$

Hence, for some $m_i \in M$, we have

$$t^h f(q_i) = t^k m_i \text{ in } M.$$

Thus,

$$t^h (f(q_i) - t^{k-h} m_i) = 0 \text{ in } M.$$

By the definition of f , the elements $f(q_i) - t^{k-h} m_i$ belong to $\mathfrak{a}^{n-h} M \subseteq \mathfrak{a}^r M$. By the definition of r , we have

$$f(q_i) = t^{k-h} m_i \text{ in } M.$$

And thus

$$\mu \circ \bar{f}(q_i) = 0 \text{ in } \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M.$$

□

Corollary 22. Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, and M is an A -module. Let r, n, k , and h be natural numbers such that

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$;
2. $k > n$ and $n > r + h$.

Assume also that the following diagram with solid arrows only is given

$$\begin{array}{ccccc} \mathfrak{a}^n M / \mathfrak{a}^k M & \xrightarrow{\nu} & \mathfrak{a}^{n-h} M / \mathfrak{a}^k M & \xrightarrow{\mu} & \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M \\ & & \uparrow g & & \nearrow t^h g' \\ A^d & \xrightarrow{Q} & A^p & & \end{array}$$

where ν and μ are the natural maps, and we have $gQ = 0$. Then there exists a homomorphism

$$g': A^p \rightarrow \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M$$

such that $g'Q = 0$ and $\mu \circ \nu \circ g = t^h g'$.

Proof. We should take $N = 0$ in Lemma 21. Just note that, in this case, $c = 0$. \square

Lemma 23. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, we are given the following exact sequence of A -modules*

$$0 \rightarrow R \rightarrow M \xrightarrow{\varphi} N \rightarrow 0$$

and P is a finite A -module such that $\mathfrak{a}^h \subseteq H_P$. Let r and c be integral numbers such that

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$;
2. $\mathfrak{a}^n M \cap R = \mathfrak{a}^{n-c}(\mathfrak{a}^c M \cap R)$ for all $n \geq c$.

Let k and n be a pair of integers such that

$$k > n, \quad n > c + h, \quad n > r + h.$$

On the following diagram, $\varphi_{n,k}$ is the map induced by φ and $R_{n,k}$ is its kernel

$$0 \rightarrow R_{n,k} \rightarrow \mathfrak{a}^n M / \mathfrak{a}^k M \xrightarrow{\varphi_{n,k}} \mathfrak{a}^n N / \mathfrak{a}^k N \rightarrow 0$$

Then, on the following diagram, for every homomorphism ψ , there exists a homomorphism ψ' such that the diagram is commutative

$$\begin{array}{ccc} \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M & \xrightarrow{\varphi_{n-h,k-h}} & \mathfrak{a}^{n-h} N / \mathfrak{a}^{k-h} N \\ \uparrow \psi' & & \uparrow \\ P & \xrightarrow{\psi} & \mathfrak{a}^n N / \mathfrak{a}^k N \end{array}$$

Proof. We take an arbitrary finite resolution of P of the form

$$\dots \rightarrow A^d \xrightarrow{Q} A^p \rightarrow A^q \rightarrow P \rightarrow 0$$

Then, the homomorphism ψ extends to a homomorphism of the resolution as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{n-h,k-h} & \xrightarrow{i} & \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M & \longrightarrow & \mathfrak{a}^{n-h} N / \mathfrak{a}^{k-h} N \longrightarrow 0 \\ & & \uparrow \mu & & \uparrow \bar{\mu} & & \uparrow \\ 0 & \longrightarrow & R_{n-h,k} & \longrightarrow & \mathfrak{a}^{n-h} M / \mathfrak{a}^k M & \longrightarrow & \mathfrak{a}^{n-h} N / \mathfrak{a}^k N \longrightarrow 0 \\ & & \uparrow \nu & & \uparrow \bar{\nu} & & \uparrow \\ & & \uparrow t^h g' & & \uparrow \bar{g} & & \uparrow \psi' \\ 0 & \longrightarrow & R_{n,k} & \longrightarrow & \mathfrak{a}^n M / \mathfrak{a}^k M & \longrightarrow & \mathfrak{a}^n N / \mathfrak{a}^k N \longrightarrow 0 \\ & & \uparrow g & & \uparrow f & & \uparrow \psi \\ A^d & \xrightarrow{Q} & A^p & \longrightarrow & A^q & \longrightarrow & P \longrightarrow 0 \end{array}$$

On this diagram, arrows μ , ν , $\bar{\mu}$, and $\bar{\nu}$ are the natural maps. Now, we will produce the dotted arrows.

By Lemma 21, there is a homomorphism $g': A^p \rightarrow R_{n-h,k-h}$ such that $g' \circ Q = 0$ and $\mu \circ \nu \circ g = t^h g'$. Since $t^h \in H_P$, the homomorphism $\mu \circ \nu \circ g$ lifts to a homomorphism $\bar{g}: A^p \rightarrow R_{n-h,k-h}$. Now, the homomorphism

$$\bar{\mu} \circ \bar{\nu} \circ f - i \circ \bar{g}: A^p \rightarrow \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M$$

induces the required one $\psi': P \rightarrow \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M$. \square

Lemma 24. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, M is an A -module, and P is a finite A -module such that $\mathfrak{a}^h \subseteq H_P$. Let r , n , and k be natural numbers such that*

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$;
2. $k > n + h$ and $n > r + h$.

Then, on the following diagram, for every homomorphism ψ , there exists a homomorphism ψ' such that the diagram is commutative

$$\begin{array}{ccc} M / \mathfrak{a}^{k-h} M & \longrightarrow & M / \mathfrak{a}^{n-h} M \\ \uparrow \scriptstyle \psi' & & \uparrow \\ P & \xrightarrow{\psi} & M / \mathfrak{a}^n M \end{array}$$

Proof. We take an arbitrary free resolution of P as follows

$$\dots \rightarrow A^d \xrightarrow{Q} A^p \rightarrow A^q \rightarrow P \rightarrow 0$$

Then the homomorphism ψ lifts to a homomorphism of the resolution as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M & \xrightarrow{i} & M / \mathfrak{a}^{k-h} M & \longrightarrow & M / \mathfrak{a}^{n-h} M & \longrightarrow & 0 \\ & & \uparrow \scriptstyle \nu & & \uparrow \scriptstyle \bar{\nu} & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{a}^n M / \mathfrak{a}^k M & \longrightarrow & M / \mathfrak{a}^k M & \longrightarrow & M / \mathfrak{a}^n M & \longrightarrow & 0 \\ & & \uparrow \scriptstyle g & & \uparrow \scriptstyle f & & \uparrow \scriptstyle \psi' & & \\ A^d & \xrightarrow{Q} & A^p & \longrightarrow & A^q & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

$\begin{array}{c} \nearrow \scriptstyle t^h g' \\ \searrow \scriptstyle \varphi \end{array}$

On this diagram, ν and $\bar{\nu}$ are the natural maps. Now, We will produce the dotted arrows.

By Corollary 22, there is a homomorphism

$$g': A^p \rightarrow \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M$$

such that $\nu \circ g = t^h g'$ and $g' \circ Q = 0$. Since $t^h \in H_P$, the homomorphism $\nu \circ g$ lifts to a homomorphism $\varphi: A^q \rightarrow \mathfrak{a}^{n-h} M / \mathfrak{a}^{k-h} M$. Then the homomorphism

$$\bar{\nu} \circ f - i \circ \varphi: A^p \rightarrow M / \mathfrak{a}^{k-h} M$$

induces the required lifting $\psi': P \rightarrow M / \mathfrak{a}^{k-h} M$. \square

Theorem 25. Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, M is an \mathfrak{a} -adically complete A -module, and P is a finite A -module such that $\mathfrak{a}^h \subseteq H_P$. Let r and n be natural numbers such that

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$;
2. $n > \max(r + h, 2h)$.

Then, on the following diagram, for every homomorphism ψ , there exists a homomorphism ψ' such that the diagram is commutative

$$\begin{array}{ccc} M & \longrightarrow & M/\mathfrak{a}^{n-h}M \\ \uparrow \scriptstyle \psi' & & \uparrow \\ P & \xrightarrow{\psi} & M/\mathfrak{a}^nM \end{array}$$

Proof. By Lemma 24, we see that, for an arbitrary $n > \max(r + h, 2h)$ and every homomorphism ψ as above, there is a homomorphism

$$\psi': P \rightarrow M/\mathfrak{a}^{2(n-h)}M$$

such that the following diagram is commutative

$$\begin{array}{ccc} M/\mathfrak{a}^{2(n-h)}M & \longrightarrow & M/\mathfrak{a}^{n-h}M \\ \uparrow \scriptstyle \psi' & & \uparrow \\ P & \xrightarrow{\psi} & M/\mathfrak{a}^nM \end{array}$$

Then, since M is complete and $2(n - h) > n$, we get the result by the induction on n . □

3.2 Lifting of ring homomorphisms

Theorem 26. Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal such that A is \mathfrak{a} -adically complete. Let $\pi: B \rightarrow \bar{B}$ be a surjective A -homomorphism of formally finitely generated A -algebras, C and \bar{C} are A -algebras such that

1. There is an exact sequence of A -algebras

$$0 \rightarrow I \rightarrow C \xrightarrow{\varphi} \bar{C} \rightarrow 0;$$

2. C is $\mathfrak{a}C$ -adically complete;

Suppose that we are given natural numbers r , h , c , and n such that

1. $\mathfrak{a}^r C \cap \text{ann}_C(t) = 0$ and $\mathfrak{a}^r \bar{C} \cap \text{ann}_{\bar{C}}(t) = 0$;

2. $\mathfrak{a}^k C \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c C \cap I)$ for all $k \geq c$;
3. $\mathfrak{a}^h \subseteq \bar{H}_{B/A} \subseteq B$ and $\mathfrak{a}^h \subseteq \bar{H}_{\bar{B}/A} \subseteq \bar{B}$;
4. $n > \max(r + 2h, c + 2h, 4h)$.

Let us be given the following commutative diagram

$$\begin{array}{ccc}
 C/\mathfrak{a}^n C & \xrightarrow{\varphi_n} & \bar{C}/\mathfrak{a}^n \bar{C} \longrightarrow 0 \\
 \alpha_n \uparrow & & \uparrow \beta_n \\
 B & \xrightarrow{\pi} & \bar{B}
 \end{array}$$

where φ_n is induced by φ .

Then, there exist A -homomorphisms $\alpha: B \rightarrow C$ and $\beta: \bar{B} \rightarrow \bar{C}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
 B & \xrightarrow{\quad \alpha \quad} & C & & \\
 \alpha_n \downarrow & \searrow \pi & \downarrow & \searrow \varphi & \\
 & & \bar{B} & \xrightarrow{\quad \beta \quad} & \bar{C} \\
 & & \downarrow \beta_n & & \downarrow \\
 C/\mathfrak{a}^n C & \xrightarrow{\quad} & C/\mathfrak{a}^{n-2h} C & \xrightarrow{\quad \varphi_{n-2h} \quad} & \bar{C}/\mathfrak{a}^{n-2h} \bar{C} \\
 & \searrow \varphi_n & & & \downarrow \\
 & & \bar{C}/\mathfrak{a}^n \bar{C} & \xrightarrow{\quad} & \bar{C}/\mathfrak{a}^{n-2h} \bar{C}
 \end{array}$$

Proof. By Lemma 1 of [3], one finds A -homomorphisms

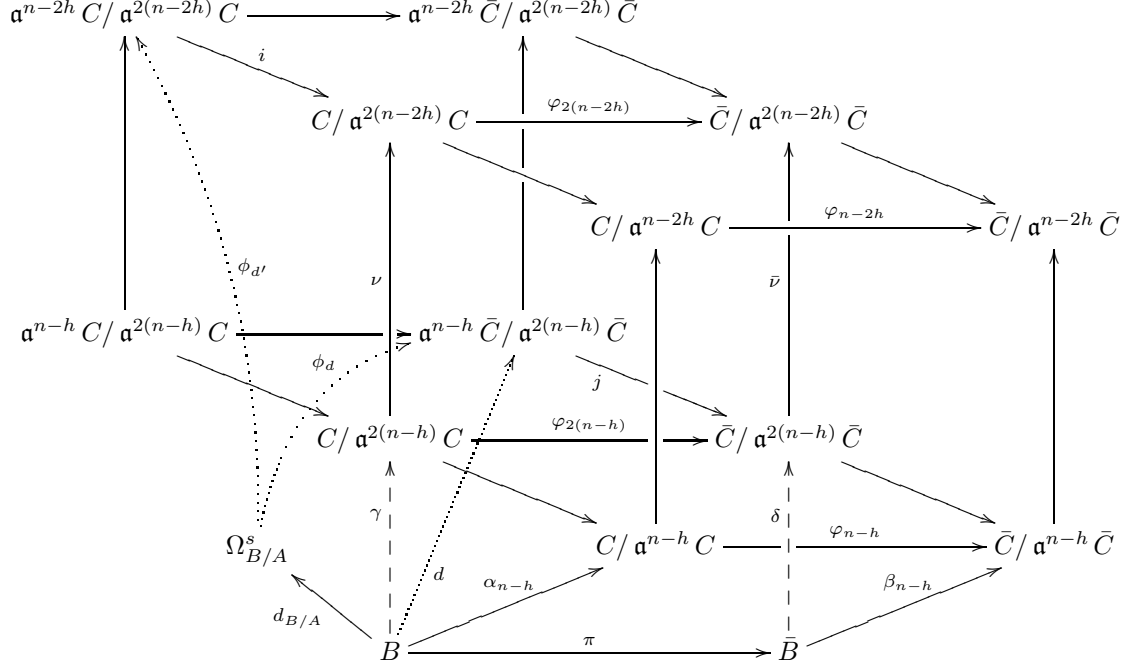
$$\gamma: B \rightarrow C/\mathfrak{a}^{2(n-h)} C \quad \text{and} \quad \delta: \bar{B} \rightarrow \bar{C}/\mathfrak{a}^{2(n-h)} \bar{C}$$

such that the following diagrams are commutative

$$\begin{array}{ccc}
 B & \xrightarrow{\quad \gamma \quad} & C/\mathfrak{a}^{2(n-h)} C \\
 \alpha_n \downarrow & \searrow \alpha_{n-h} & \downarrow \\
 C/\mathfrak{a}^n C & \xrightarrow{\quad} & C/\mathfrak{a}^{n-h} C
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{B} & \xrightarrow{\quad \delta \quad} & \bar{C}/\mathfrak{a}^{2(n-h)} \bar{C} \\
 \beta_n \downarrow & \searrow \beta_{n-h} & \downarrow \\
 \bar{C}/\mathfrak{a}^n \bar{C} & \xrightarrow{\quad} & \bar{C}/\mathfrak{a}^{n-h} \bar{C}
 \end{array}$$

Where, α_{n-h} and β_{n-h} are the compositions of α_n and β_n with the corresponding quotient map, respectively.

We will draw everything on the following diagram



On this diagram $i, j, \nu, \bar{\nu}$ are the natural maps. We will produce all dotted arrows step by step. Before this, let us note that the diagram is commutative only for the solid arrows. The four lines of this diagram (going from the back facet to the front facet) are the extensions of rings by modules. Therefore, all modules on the back facet are B -modules.

By construction, the difference of homomorphisms

$$\varphi_{2(n-h)} \circ \gamma - \delta \circ \pi$$

factors through j , that is, equals $j \circ d$ for some

$$d: B \rightarrow \mathfrak{a}^{n-h}\bar{C}/\mathfrak{a}^{2(n-h)}\bar{C}$$

and d is a derivation of B over A . The derivation d corresponds to a B -homomorphism ϕ_d . Then, by Lemma 23, we find a B -homomorphism

$$\phi_{d'}: \Omega_{B/A}^s \rightarrow \mathfrak{a}^{n-2h}C/\mathfrak{a}^{2(n-2h)}C.$$

This homomorphism corresponds to the derivation $d' = \phi_{d'} \circ d_{B/A}$, where

$$d': B \rightarrow \mathfrak{a}^{n-2h}C/\mathfrak{a}^{2(n-2h)}C.$$

Thus, the map

$$\gamma' = \nu \circ \gamma - i \circ d': B \rightarrow C/\mathfrak{a}^{2(n-2h)}C$$

is a homomorphism and γ' together with

$$\bar{\nu} \circ \delta: \bar{B} \rightarrow \bar{C}/\mathfrak{a}^{2(n-2h)} \bar{C}$$

gives the required lifting. Now, since C and, hence, \bar{C} are \mathfrak{a} -adically complete, the result follows by induction on n . \square

Remark 27. In the proof of Theorem 26, we cannot cite Lemma 1 of [3] because they use the ideal H_J instead of $\bar{H}_{B/A}$. There are two possible solutions of this issue. The first one is to see that the proof of Lemma 1 is word by word applicable for the ideal $\bar{H}_{B/A}$. The second way is to use Lemma 12 saying that, for some natural l , we have $\bar{H}_{B/A}^l \subseteq H_J$. Then $\mathfrak{a}^{hl} \subseteq H_J$. However, in this case, we should use hl instead of h .

Theorem 28. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ a principal ideal. Let $\phi: B \rightarrow \bar{B}$ be a surjective A -homomorphism of finitely generated A -algebras, C and \bar{C} are A -algebras such that*

1. B and \bar{B} are smooth over the complement of $V(\mathfrak{a})$;
2. There is an exact sequence of A -algebras

$$0 \rightarrow I \rightarrow C \xrightarrow{\varphi} \bar{C} \rightarrow 0;$$

3. The pairs $(C, \mathfrak{a}C)$ and $(\bar{C}, \mathfrak{a}\bar{C})$ are henselian.

Suppose we are given homomorphisms $\varepsilon: \hat{B} \rightarrow \hat{C}$ and $\bar{\varepsilon}: \hat{\bar{B}} \rightarrow \hat{\bar{C}}$ such that the following diagram with the solid arrows only is commutative

$$\begin{array}{ccccc}
 & & \hat{C} & \xleftarrow{\varepsilon} & \hat{B} \\
 & \swarrow & \uparrow \bar{\varepsilon} & & \swarrow \hat{\phi} \\
 \hat{\bar{C}} & \xleftarrow{\quad} & \hat{\bar{B}} & & \\
 \uparrow & & \uparrow & & \uparrow \\
 0 & \swarrow & 0 & & \\
 & \swarrow & \uparrow & & \swarrow \hat{\phi} \\
 & & C & \xleftarrow{\gamma} & B \\
 & \swarrow & \uparrow & & \swarrow \phi \\
 \bar{\bar{C}} & \xleftarrow{\quad} & \bar{\bar{B}} & & \\
 \uparrow & & \uparrow & & \uparrow \\
 0 & \swarrow & 0 & & \\
 & \swarrow & \uparrow & & \swarrow \phi \\
 & & \bar{C} & \xleftarrow{\bar{\gamma}} & \bar{B}
 \end{array}$$

Then, for every natural number n , there exist two homomorphisms $\gamma: B \rightarrow C$ and $\bar{\gamma}: \bar{B} \rightarrow \bar{C}$ such that the bottom square on the previous diagram is commutative and the following diagrams are commutative

$$\begin{array}{ccc}
 & \hat{C} & \\
 \hat{B} \xrightarrow{\varepsilon} & & \hat{C}/\mathfrak{a}^n \hat{C} \\
 \hat{B} \xrightarrow{\bar{\gamma}} & \hat{C} & \uparrow \\
 & \hat{C} & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 & \hat{C} & \\
 \hat{\bar{B}} \xrightarrow{\bar{\varepsilon}} & & \hat{\bar{C}}/\mathfrak{a}^n \hat{\bar{C}} \\
 \hat{\bar{B}} \xrightarrow{\bar{\gamma}} & \hat{\bar{C}} & \uparrow \\
 & \hat{\bar{C}} &
 \end{array}$$

Proof. We fix some generators of B and get the following exact sequences

$$\begin{aligned} 0 \rightarrow J \rightarrow A[x] \xrightarrow{\beta} B \rightarrow 0 \\ 0 \rightarrow \bar{J} \rightarrow A[x] \xrightarrow{\bar{\beta}} \bar{B} \rightarrow 0 \end{aligned}$$

where $x = \{x_1, \dots, x_n\}$ is a finite set of indeterminates.

Now, we replace B by $S_B(J/J^2)$ and replace ϕ by the composition of the projection $S_B(J/J^2) \rightarrow B$ (sending J/J^2 to zero) and ϕ . So, we may assume that $(\Omega_{B/A})_t$ is free B_t -module of rank d . Adding the generators y of J/J^2 to the generators of B , we get a surjective homomorphism

$$\beta': A[x, y] \rightarrow S_B(J/J^2).$$

Let $t = \{t_1, \dots, t_d\}$ be the set of indeterminates, we replace β by the composition of the projection

$$A[x, y, t] \rightarrow A[x, y], \quad t \mapsto 0$$

and β' . The homomorphism $\bar{\beta}$ will be replaced by the composition of the projection

$$A[x, y, t] \rightarrow A[x], \quad y, t \mapsto 0$$

and $\bar{\beta}$. Now, J is the kernel of the new homomorphism β . By Lemma 3 of [3], we see that $(J/J^2)_t$ is a free B_t -module.

Let the elements $(f_1, \dots, f_q) \subseteq J$ form a basis of $(J/J^2)_t$. Then there is an element $e \in J$ such that

$$e^2 - t^h e \in (f_1, \dots, f_q)$$

for some natural h . Since the sequence

$$0 \rightarrow (J/J^2)_t \rightarrow \Omega_{A[x, y, t]/A} \otimes_{A[x, y, t]} B_t \rightarrow \Omega_{B_t/A} \rightarrow 0$$

is split exact, the ideal

$$\Delta^q(f_1, \dots, f_q) \subseteq A[x, y, t]$$

generated by q -minors of the Jacobian matrix of f_1, \dots, f_q contains some $t^{h'}$. Enlarging h' or replacing e by $t^{h-h'}e$, we may assume that $h = h'$.

By the Artin-Rees lemma, there are natural numbers r and c such that

$$\begin{aligned} \mathfrak{a}^r C \cap \text{ann}_C(t) &= 0 \\ \mathfrak{a}^m C \cap I &= \mathfrak{a}^{m-c}(\mathfrak{a}^c C \cap I) \quad \text{whenever } m \geq c \end{aligned}$$

Now, we define the ideal

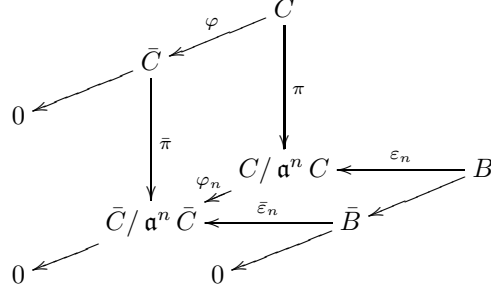
$$J_0 = (f_1, \dots, f_q) \subseteq A[x, y, t]$$

and the element $s = (t^h - e)t^h$. Since

$$J_t = (f_1, \dots, f_q, e)_t \quad \text{and} \quad e^2 - t^h e \in (f_1, \dots, f_q),$$

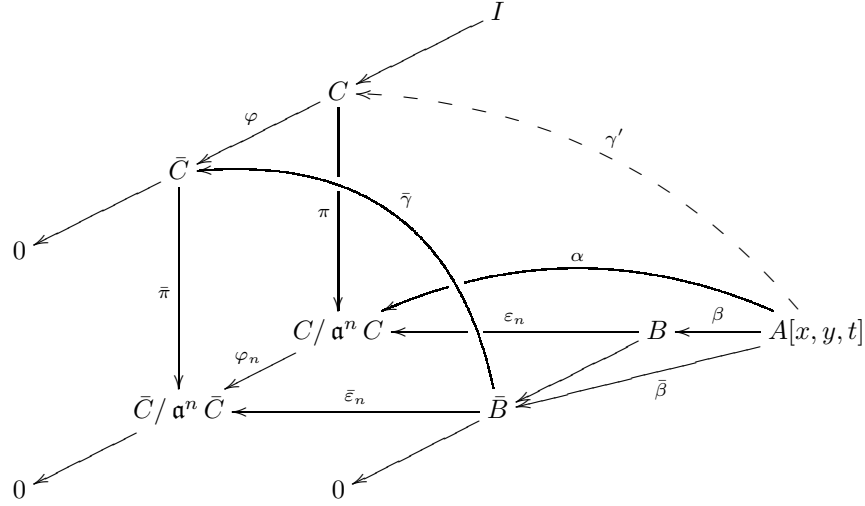
we have $J_s = (f_1, \dots, f_q)_s$. Therefore, $s^p J \subseteq J_0$ for some natural p .

Now, we fix a natural number $n > \max(2h+c, h+r)$ and take the reductions of ε and $\bar{\varepsilon}$ modulo $\mathfrak{a}^n C$ and $\mathfrak{a}^n \bar{C}$, respectively:



Here π and $\bar{\pi}$ are the quotient maps, φ_n is induced by φ , and the homomorphisms ε_n and $\bar{\varepsilon}_n$ are the reductions of ε and $\bar{\varepsilon}$, respectively.

Now, by Theorem 2 bis of [3], the homomorphism $\bar{\varepsilon}$ approximates by some $\bar{\gamma}: \bar{B} \rightarrow \bar{C}$ module $\mathfrak{a}^n \bar{C}$. We denote the composition $\varepsilon_n \circ \beta$ by α and let γ' be an arbitrary lifting of $\bar{\gamma} \circ \bar{\beta}$. We draw everything on the following diagram:



The map γ' is denoted by the dotted line because the triangle with γ' , α , and π is not necessarily commutative. Firstly, we are going to replace γ' by γ'' such that $\pi \circ \gamma'' = \alpha$. Secondly, we will show how to replace γ'' by γ such that $\gamma(J) = 0$.

Let us define the following vector

$$k' = \pi \circ \gamma'(x, y, t) - \alpha(x, y, t).$$

If $k' = (k'_i)$, then $k'_i \in \ker \varphi_n$. Therefore, we can find $k = (k_i)$ with $k_i \in I$ and such that $\pi(k_i) = k'_i$. Let us define the following homomorphism of A -algebras

$$\gamma'': A[x, y, t] \rightarrow C \quad \text{where} \quad (x, y, t) \mapsto \gamma'(x, y, t) - k.$$

Since γ'' coincides with γ' modulo I the whole diagram with γ'' instead of γ' is commutative. Now, we are going to show that $\gamma(J) = 0$. We will do this in two steps.

Step 1. We will show that there is a homomorphism $\gamma: A[x, y, t] \rightarrow C$ such that $\gamma(J_0) = 0$. Let $a = \gamma''(x, y, t)$ be a vector of elements of C . We will search the homomorphism γ among ones sending (x, y, t) to $a + t^h p$ for some vector $p = (p_i)$, where $p_i \in \mathfrak{a}^{n-2h} C \cap I$. We also denote $f = (f_i)$, the vector of generators of J_0 .

By construction, we have

$$\gamma''(J_0) \subseteq \gamma''(J) \subseteq \mathfrak{a}^n C \cap I = \mathfrak{a}^{n-c}(\mathfrak{a}^c C \cap I)$$

Thus $f(a) = t^{2h}e$, where $e = (e_i)$ and $e_i \in \mathfrak{a}^{n-2h} C \cap I$. Since $t^h \in \Delta^q(f)$, we have $M(a)N = t^h E$, where $M(a)$ is the Jacobian matrix of f at the point a , N is some matrix and E is the identity matrix. Now, we want to have

$$0 = f(a + tp) = f(a) + t^h M(a)p + t^{2h} p^t Q p.$$

where the right-hand part is the Taylor expansion up to the second order term and p^t is the transposed vector p . We replace $f(a)$ by $t^{2h}e$ and $t^h E$ by $M(a)N$. As the result, we will obtain the following equation

$$0 = t^h M(a)(p + N(e + p^t Q p))$$

It is enough to find $p \in \mathfrak{a}^{n-2h} C \cap I$ such that

$$p + N(e + p^t Q p) = 0.$$

The zero vector $p = 0$ is a solution of this system in $C/\mathfrak{a}^{n-2h} C$ and the Jacobian matrix of this system at $p = 0$ is the identity matrix E . Since $(C, \mathfrak{a} C)$ is henselian, there is a solution p in C such that $p_i \in \mathfrak{a}^{n-2h} C$. But then

$$(E + p^t Q)p = -Ne, \quad \text{and} \quad e_i \in \mathfrak{a}^{n-2h} C \cap I$$

Since all p_i are in the radical of C , the matrix $(E + p^t Q)$ is invertible and, hence, $p_i \in \mathfrak{a}^{n-2h} C \cap I$.

Step 2. We will show that $\gamma(J) = 0$. By construction, the completion of the map

$$\gamma: A[x, y, t] \rightarrow C \quad \text{by} \quad (x, y, t) \mapsto a + t^h p$$

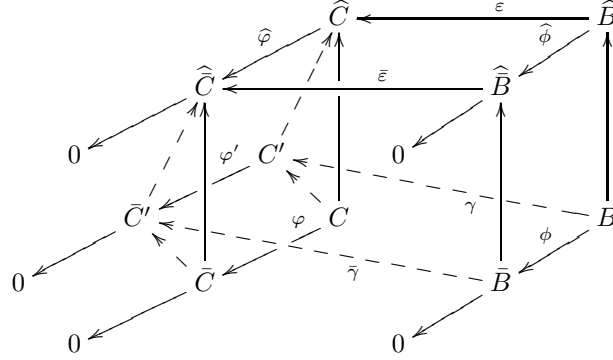
coincides with ε modulo $\mathfrak{a}^{n-h} \widehat{C}$. Thus $\gamma(J) \subseteq \mathfrak{a}^{n-h} C$. Moreover, by definition of s , we have

$$t^{2hp} \gamma(J) = \gamma(s^p J) \subseteq \gamma(J_0) = 0$$

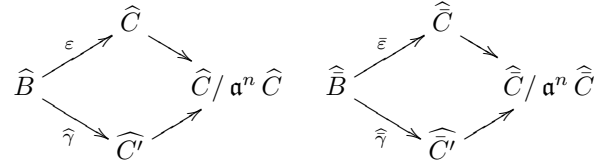
Then by the choice of n and r , it follows that $\gamma(J) = 0$.

We have constructed $\gamma: A[x, y, t] \rightarrow C$ being a lifting of $\varepsilon_{n-h} \circ \beta$. It induces a homomorphism of $\gamma: B \rightarrow C$ being a lifting of ε_{n-h} . Since $\bar{\gamma}$ is a lifting of $\bar{\varepsilon}_n$, it is also a lifting of $\bar{\varepsilon}_{n-h}$. Since n is an arbitrary large, we are done. \square

Corollary 29. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ a principal ideal. Let $\phi: B \rightarrow \bar{B}$ be a surjective A -homomorphism of finitely generated A -algebras, $\varphi: C \rightarrow \bar{C}$ is a homomorphism of A -algebras. Assume that B and \bar{B} are smooth over the complement of $V(\mathfrak{a})$ and we are given homomorphisms $\varepsilon: \hat{B} \rightarrow \hat{C}$ and $\bar{\varepsilon}: \hat{\bar{B}} \rightarrow \hat{\bar{C}}$ such that the following diagram with the solid arrows only is commutative*



Then, for every natural number n , there exist étale extensions $C \rightarrow C'$ and $\bar{C} \rightarrow \bar{C}'$ such that the induced by φ map $\varphi': C' \rightarrow \bar{C}'$ is well-defined and surjective. There also exist two homomorphisms $\gamma: B \rightarrow C$ and $\bar{\gamma}: \bar{B} \rightarrow \bar{C}$ such that the squares with dotted arrows on the previous diagram is commutative, and the following diagrams are commutative



Lemma 30. *Let A be a ring, $\mathfrak{a} \subseteq A$, A is \mathfrak{a} -adically complete, and $M \subseteq P$ are finitely generated A -modules. Then $M = \bigcap_{k \geq 0} M + \mathfrak{a}^k P$.*

Proof. Since A is complete, the module P/M is complete. In particular, P/M is separated. Thus $\bigcap_{k \geq 0} \mathfrak{a}^k P/M = 0$. Whence $M = \bigcap_{k \geq 0} M + \mathfrak{a}^k P$. \square

The following corollary is a direct generalization of Theorem 2 bis of [3]. The difference between the corollary and Theorem 28 is the additional condition on the sections. They must factors through given open subsets.

Corollary 31. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal,*

$$\phi: B \rightarrow \bar{B} \quad \text{and} \quad \psi: C \rightarrow \bar{C}$$

be surjective homomorphisms of A -algebras such that B and \bar{B} are finitely generated, and the pairs $(C, \mathfrak{a}C)$ and $(\bar{C}, \mathfrak{a}\bar{C})$ are henselian. Let us be given ideals $I \subseteq B$, $\bar{I} \subseteq \bar{B}$ and A -homomorphisms

$$\varphi: B \rightarrow \hat{C} \quad \text{and} \quad \bar{\varphi}: \bar{B} \rightarrow \hat{\bar{C}}$$

such that

$$\varphi(I)\widehat{C} \supseteq \mathfrak{a}^c \quad \text{and} \quad \bar{\varphi}(\bar{I})\widehat{\bar{C}} \supseteq \mathfrak{a}^c$$

for some natural c . Then, for every n , there exist A -homomorphisms

$$\gamma: B \rightarrow C \quad \text{and} \quad \bar{\gamma}: \bar{B} \rightarrow \bar{C}$$

such that

$$\gamma(I)C \supseteq \mathfrak{a}^c \quad \text{and} \quad \bar{\gamma}(\bar{I})\bar{C} \supseteq \mathfrak{a}^c$$

and the following diagram is commutative

$$\begin{array}{ccccc}
 & C & \xleftarrow{\quad \gamma \quad} & B & \\
 \psi \swarrow & \downarrow & \xleftarrow{\quad \bar{\gamma} \quad} & \downarrow \phi & \\
 \bar{C} & & \bar{B} & & \\
 \downarrow & \swarrow \psi_n & \downarrow \bar{\varphi} & \swarrow \hat{\psi} & \\
 \bar{C}/\mathfrak{a}^n \bar{C} & \leftarrow C/\mathfrak{a}^n C & \leftarrow \widehat{C} & & \\
 & & \downarrow & & \\
 & & \widehat{\bar{C}} & &
 \end{array}$$

Proof. We fix $m > \max(n, c)$ and apply Theorem 28 with m instead of n . Therefore, we find A -homomorphisms γ and $\bar{\gamma}$ as on the previous diagram. We will show that $\mathfrak{a}^c \subseteq \gamma(I)C$, the second inclusion is proven in the same way. By definition, $\varphi(I) \subseteq \gamma(I) + \mathfrak{a}^m \widehat{C}$. Hence $(\mathfrak{a} \widehat{C})^c \subseteq \gamma(I)\widehat{C} + (\mathfrak{a} \widehat{C})^m$. Thus, by induction, we have $(\mathfrak{a} \widehat{C})^c \subseteq \gamma(I)\widehat{C} + (\mathfrak{a} \widehat{C})^k$ for all $k \geq m$. By Lemma 30, $\mathfrak{a}^c \widehat{C} \subseteq \gamma(I)\widehat{C}$. Since C is henselian, $C \rightarrow \widehat{C}$ is faithfully flat. Therefore, for any ideal $J \subseteq C$, we have $(J\widehat{C}) \cap C = J$. In particular, $\mathfrak{a}^c C \subseteq \gamma(I)C$. \square

4 Approximation of pairs

4.1 Approximation of henselizations

Theorem 32. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ is a principal ideal, $\phi: B \rightarrow \bar{B}$ and $\psi: C \rightarrow \bar{C}$ are surjective A -homomorphisms of A -algebras such that*

1. B and \bar{B} are finitely generated over A .
2. $(C, \mathfrak{a}C)$ and $(\bar{C}, \mathfrak{a}\bar{C})$ are henselian.
3. I is the kernel of ψ .

Let n, r, h, c be natural numbers such that

1. $\mathfrak{a}^r C \cap \text{ann}_C(t) = 0$ and $\mathfrak{a}^r \bar{C} \cap \text{ann}_{\bar{C}}(t) = 0$.
2. $\mathfrak{a}^k C \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c C \cap I)$ for all $k \geq c$.

3. $\mathfrak{a}^h \subseteq H_{B/A}$ and $\mathfrak{a}^h \subseteq H_{\bar{B}/A}$.

4. $n > \max(r + 2h, c + 2h, 4h)$.

and we are given a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \bar{B} \\ \downarrow \alpha_n & & \downarrow \beta_n \\ C/\mathfrak{a}^n C & \xrightarrow{\psi_n} & \bar{C}/\mathfrak{a}^n \bar{C} \end{array}$$

Then there exist A -homomorphisms $\alpha: B \rightarrow C$ and $\beta: \bar{B} \rightarrow \bar{C}$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & C & \xleftarrow{\quad \alpha \quad} & B \\ & \swarrow \psi & \downarrow & \nwarrow \phi & \downarrow \alpha_n \\ \bar{C} & \xleftarrow{\quad \beta \quad} & \bar{B} & & \\ & \downarrow & \downarrow \beta_n & & \\ & C/\mathfrak{a}^{n-2h} C & \xleftarrow{\quad \beta_n \quad} & C/\mathfrak{a}^n C & \\ & \swarrow \psi_{n-2h} & \downarrow & \swarrow \psi_n & \\ \bar{C}/\mathfrak{a}^{n-2h} \bar{C} & \xleftarrow{\quad \quad \quad} & \bar{C}/\mathfrak{a}^n \bar{C} & & \end{array}$$

Proof. The assertion immediately follows from Theorem 26 and Theorem 28. \square

Corollary 33. Let A be a ring $\mathfrak{a} = (t) \subseteq A$ is a principal ideal, $\phi: B \rightarrow \bar{B}$ and $\psi: C \rightarrow \bar{C}$ are surjective A -homomorphisms of finitely generated A -algebras, and I is the kernel of ψ . Let n, r, h, c be natural numbers such that

1. $\mathfrak{a}^r C \cap \text{ann}_C(t) = 0$ and $\mathfrak{a}^r \bar{C} \cap \text{ann}_{\bar{C}}(t) = 0$.
2. $\mathfrak{a}^k C \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c C \cap I)$ for all $k \geq c$.
3. $\mathfrak{a}^h \subseteq H_{B/A}$ and $\mathfrak{a}^h \subseteq H_{\bar{B}/A}$.
4. $n > \max(r + 2h, c + 2h, 4h)$.

and we are given a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \bar{B} \\ \downarrow \alpha_n & & \downarrow \beta_n \\ C/\mathfrak{a}^n C & \xrightarrow{\psi_n} & \bar{C}/\mathfrak{a}^n \bar{C} \end{array}$$

Then there exist strict etale extensions $C \rightarrow C'$, $\bar{C} \rightarrow \bar{C}'$, and a surjective A -homomorphism $\psi': C' \rightarrow \bar{C}'$ together with A -homomorphisms $\alpha: B \rightarrow C'$ and

$\beta: \bar{B} \rightarrow \bar{C}'$ such that the following diagram is commutative

$$\begin{array}{ccccc}
& & C' & \xleftarrow{\alpha} & B \\
& \swarrow \psi' & \downarrow & \swarrow \phi & \downarrow \alpha_n \\
\bar{C}' & \xleftarrow{\beta} & \bar{B} & & \\
\downarrow & & \downarrow & \xleftarrow{\beta_n} & \downarrow \\
\bar{C}/\mathfrak{a}^{n-2h}\bar{C} & \xleftarrow{\psi_{n-2h}} & C/\mathfrak{a}^{n-2h}C & \xleftarrow{\psi_n} & C/\mathfrak{a}^n C
\end{array}$$

Lemma 34. Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ an ideal, $\alpha: A \rightarrow B$ is a homomorphism such that $B/\mathfrak{a}^k B$ is a finitely generated A/\mathfrak{a}^k -algebra for all k . Let us be given numbers $n > r$ such that

1. $\mathfrak{a}^r B \cap \text{ann}_B(t) = 0$.
2. The map $\alpha_n: A/\mathfrak{a}^n \rightarrow B/\mathfrak{a}^n B$ is smooth.

Assume that $\mathfrak{a} B$ belongs to the Jacobson radical of B , then all the maps α_k are smooth.

Proof. By Proposition 17.5.2 of [6], it is enough to show that all the maps α_k are flat. Since α_n is flat, the maps

$$\mathfrak{a}^k / \mathfrak{a}^{k+1} \otimes_{A/\mathfrak{a}} B/\mathfrak{a} B \rightarrow \mathfrak{a}^k B / \mathfrak{a}^{k+1} B$$

are isomorphisms for all $k < n$ by Theorem 49 of [7]. Because of the choice of r , the maps

$$\mathfrak{a}^r / \mathfrak{a}^{r+1} \xrightarrow{t^{k-r}} \mathfrak{a}^k / \mathfrak{a}^{k+1} \quad \text{and} \quad \mathfrak{a}^r B / \mathfrak{a}^{r+1} B \xrightarrow{t^{k-r}} \mathfrak{a}^k B / \mathfrak{a}^{k+1} B$$

are isomorphisms. Hence,

$$\mathfrak{a}^k / \mathfrak{a}^{k+1} \otimes_{A/\mathfrak{a}} B/\mathfrak{a} B \rightarrow \mathfrak{a}^k B / \mathfrak{a}^{k+1} B$$

holds for all k and, by Theorem 49 of [7], the maps α_k are flat for all k . \square

Lemma 35. Let A and B be rings, $\mathfrak{a} \subseteq A$ an ideal, $\alpha: A \rightarrow B$ is a homomorphism such that B becomes a localisation of a finitely generated A -algebra. Suppose also that $\mathfrak{a} B$ belongs to the Jacobson radical of B and $\alpha_k: A/\mathfrak{a}^k \rightarrow B/\mathfrak{a}^k B$ are smooth. Then α is smooth.

Proof. Since B is a localisation of a finitely generated algebra, there is an exact sequence of A -homomorphisms

$$0 \rightarrow J \rightarrow R \rightarrow B \rightarrow 0$$

where R is smooth over A . The condition on α_k implies that B is smooth over A in \mathfrak{a} - B -topology (see [7]). Thus, for all n , the homomorphism of $B/\mathfrak{a}^n B$ -modules

$$J/J^2 \otimes_B B/\mathfrak{a}^n B \rightarrow \Omega_{R/A} \otimes_R B/\mathfrak{a}^n B$$

is left-invertible by Theorem 63 of [7]. Since $\Omega_{R/A} \otimes_R B$ is projective and \mathfrak{a} belongs to the Jacobson radical of B , the homomorphism

$$J/J^2 \rightarrow \Omega_{R/A} \otimes_R B$$

is left-invertible by Lemma 2 of [7, (29.A), p. 215]. Hence, again by Theorem 63, B is smooth over A . \square

Lemma 36. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ is a principal ideal, $\phi: B \rightarrow \bar{B}$ and $\psi: C \rightarrow \bar{C}$ are surjective A -homomorphisms of finitely generated A -algebras, and I is the kernel of ψ . Let n, r, h, c be natural numbers such that*

1. $\mathfrak{a}^r C \cap \text{ann}_C(t) = 0$ and $\mathfrak{a}^r \bar{C} \cap \text{ann}_{\bar{C}}(t) = 0$.
2. $\mathfrak{a}^k C \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c C \cap I)$ for all $k \geq c$.
3. $\mathfrak{a}^h \subseteq H_{B/A}$ and $\mathfrak{a}^h \subseteq H_{\bar{B}/A}$.
4. $n > \max(r + 2h, c + 2h, 4h)$.

and we are given a commutative diagram

$$\begin{array}{ccc} B/\mathfrak{a}^n B & \xrightarrow{\phi_n} & \bar{B}/\mathfrak{a}^n \bar{B} \\ \downarrow \alpha_n & & \downarrow \beta_n \\ C/\mathfrak{a}^n C & \xrightarrow{\psi_n} & \bar{C}/\mathfrak{a}^n \bar{C} \end{array}$$

where α_n and β_n are smooth. Then there exist strict etale extensions $C \rightarrow C'$, $\bar{C} \rightarrow \bar{C}'$, and a surjective A -homomorphism $\psi': C' \rightarrow \bar{C}'$ together with smooth A -homomorphisms $\alpha: B \rightarrow C'$ and $\beta: \bar{B} \rightarrow \bar{C}'$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & C' & \xleftarrow{\quad \alpha \quad} & B \\ & \swarrow \psi' & \downarrow & \swarrow \phi & \downarrow \alpha_n \\ C' & \xleftarrow{\quad \beta \quad} & \bar{B} & & \\ \downarrow & & \downarrow & & \\ & \swarrow \psi_{n-2h} & C/\mathfrak{a}^{n-2h} C & \xleftarrow{\beta_n} & C/\mathfrak{a}^n C \\ & & \downarrow & \swarrow \psi_n & \\ \bar{C}/\mathfrak{a}^{n-2h} \bar{C} & \xleftarrow{\quad \quad \quad} & \bar{C}/\mathfrak{a}^n \bar{C} & & \end{array}$$

Proof. By Corollary 33, we find the homomorphisms α and β . We will find an element $s \in 1 + \mathfrak{a}C'$ such that the compositions

$$B \xrightarrow{\alpha} C' \rightarrow C'_s \quad \text{and} \quad \bar{B} \xrightarrow{\beta} \bar{C}' \rightarrow \bar{C}'_s$$

are smooth and this will prove the lemma.

Let us consider the case of α . We also denote C' by C and ψ' by ψ for short. Let $S = 1 + \mathfrak{a}B$ and $T = 1 + \mathfrak{a}C$, then the induced homomorphism $S^{-1}B \rightarrow T^{-1}C$ satisfies the conditions of Lemma 34. Hence, all $\alpha_k: B/\mathfrak{a}^k B \rightarrow C/\mathfrak{a}^k C$ are smooth. Thus, $T^{-1}C$ is smooth over $S^{-1}B$ by Lemma 35. Consequently, $T^{-1}C$ is B smooth. Now, we see that

$$T^{-1}(H_{C/B}) = H_{(T^{-1}C)/B} = (1)$$

by Lemma 4. Thus, there is an element $t \in T$, such that

$$H_{C_t/B} = (H_{C/B})_t = (1).$$

Hence, C_t is smooth over B .

If $\bar{t} \in 1 + \mathfrak{a}\bar{C}$ denote an element such that $\bar{C}_{\bar{t}}$ is \bar{B} -smooth, then we lift it to an element $\bar{t}' \in 1 + \mathfrak{a}\bar{C}$. Whence, $s = t\bar{t}'$ is the required element. \square

Lemma 37. *Let A be a ring, $\mathfrak{a} \subseteq A$ and ideal, and $\varphi: A \rightarrow A$ is a homomorphism such that A is \mathfrak{a} -adically complete, φ takes \mathfrak{a} to \mathfrak{a} and induces the identity map on A/\mathfrak{a} . Then φ is an isomorphism*

Proof. By definition, $\varphi(x) = x + \gamma(x)$, where $\gamma: A \rightarrow \mathfrak{a}$ is a linear map such that $\gamma(xy) = x\gamma(y) + \gamma(x)y + \gamma(x)\gamma(y)$. Hence, for any $x \in \mathfrak{a}$, $\gamma^n(x) \subseteq \mathfrak{a}^n$. Thus the map

$$\psi(x) = \sum_{n=0}^{\infty} (-1)^n \gamma^n(x)$$

is well-defined and is inverse to φ . \square

Proposition 38. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ is a principal ideal, $\phi: B \rightarrow \bar{B}$ and $\psi: C \rightarrow \bar{C}$ are surjective A -homomorphisms of finitely generated A -algebras, and I is the kernel of ψ .*

Let n, r, h, c be natural numbers such that

1. $\mathfrak{a}^r C \cap \text{ann}_C(t) = 0$ and $\mathfrak{a}^r \bar{C} \cap \text{ann}_{\bar{C}}(t) = 0$.
2. $\mathfrak{a}^k C \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c C \cap I)$ for all $k \geq c$.
3. $\mathfrak{a}^h \subseteq H_{B/A}$ and $\mathfrak{a}^h \subseteq H_{\bar{B}/A}$.
4. $n > \max(r + 2h, c + 2h, 4h)$.

and we are given a commutative diagram

$$\begin{array}{ccc} B/\mathfrak{a}^n B & \xrightarrow{\phi_n} & \bar{B}/\mathfrak{a}^n \bar{B} \\ \downarrow \alpha_n & & \downarrow \beta_n \\ C/\mathfrak{a}^n C & \xrightarrow{\psi_n} & \bar{C}/\mathfrak{a}^n \bar{C} \end{array}$$

where α_n and β_n are isomorphisms. Then there exist A -isomorphisms $\alpha^h: B^h \rightarrow C^h$ and $\beta^h: \bar{B}^h \rightarrow \bar{C}^h$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & C^h & \xleftarrow{\alpha^h} & B^h \\ & \swarrow \psi^h & \downarrow & \swarrow \phi^h & \downarrow \alpha_n \\ \bar{C}^h & \xleftarrow{\beta^h} & \bar{B}^h & & \\ \downarrow & & \downarrow & \swarrow \beta_n & \downarrow \psi_n \\ & C/\mathfrak{a}^{n-2h} C & \xleftarrow{\beta_n} & C/\mathfrak{a}^n C & \\ \downarrow \psi_{n-2h} & & \downarrow & \swarrow \psi_n & \\ \bar{C}/\mathfrak{a}^{n-2h} \bar{C} & \xleftarrow{\beta_n} & \bar{C}/\mathfrak{a}^n \bar{C} & & \end{array}$$

Proof. From Lemma 36 it follows that, replacing C and \bar{C} by their strict etale extensions, we may suppose that we have a pair of smooth homomorphisms $\alpha: B \rightarrow C$ and $\beta: \bar{B} \rightarrow \bar{C}$ such that the following diagram is commutative

$$\begin{array}{ccccc} & & C & \xleftarrow{\alpha} & B \\ & \swarrow \psi & \downarrow & \swarrow \phi & \downarrow \\ \bar{C} & \xleftarrow{\beta} & \bar{B} & & \\ \downarrow & & \downarrow & \swarrow \alpha_{n-2h} & \downarrow \psi_n \\ & C/\mathfrak{a}^{n-2h} C & \xleftarrow{\alpha_{n-2h}} & B/\mathfrak{a}^{n-2h} B & \\ \downarrow \psi_{n-2h} & & \downarrow & \swarrow \psi_n & \\ \bar{C}/\mathfrak{a}^{n-2h} \bar{C} & \xleftarrow{\beta_{n-2h}} & \bar{B}/\mathfrak{a}^{n-2h} \bar{B} & & \end{array}$$

Now, we should show that α and β induce isomorphisms of henselizations. We will demonstrate this for α .

Firstly, note that all α_n are isomorphisms by Lemma 37. Then, we may tensor by B^h over B . Hence, we may assume that $(B, \mathfrak{a} B)$ is henselian and C is a smooth finitely generated B -algebra. Thus, by Theorem 2 of [3], the composition

$$C \rightarrow C/\mathfrak{a}^{n-2h} C \xrightarrow{\alpha_{n-2h}^{-1}} B/\mathfrak{a}^{n-2h} B$$

lifts to a B -homomorphism $\varepsilon: C \rightarrow B$. We will show that the induced map $\varepsilon^h: C^h \rightarrow B^h = B$ is an isomorphism. The surjectivity is clear. Since C^h is a subalgebra of \widehat{C} , it is enough to show that $\widehat{\varepsilon}: \widehat{C} \rightarrow \widehat{B}$ is injective. But

$$(\widehat{\alpha} \circ \widehat{\varepsilon})_{n-2h} = \alpha_{n-2h} \circ \varepsilon_{n-2h} = Id$$

Therefore, by Lemma 37, $\widehat{\alpha} \circ \widehat{\varepsilon}$ is an isomorphism and we are done. \square

4.2 Approximation of complete algebras

Lemma 39. *Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal, and $\varphi: A \rightarrow B$ is a homomorphism such that A is \mathfrak{a} -adically complete, B is $\mathfrak{a}B$ -adically separated, and the map $\bar{\varphi}: A/\mathfrak{a} \rightarrow B/\mathfrak{a}B$ induced by φ is surjective. Then the map φ is surjective.*

Proof. Since $B = \text{Im } \varphi + \mathfrak{a}B$, for every $b \in B$, there exist $a_1 \in A$, $s_1 \in \mathfrak{a}$, and $b_1 \in B$ such that $b = \varphi(a_1) + s_1 b_1$. Applying the arguments for b_1 instead of b we find $a_1 \in A$, $s_2 \in \mathfrak{a}$, and $b_2 \in B$ such that $b_1 = \varphi(a_2) + s_2 b_2$. Hence

$$b = \varphi(a_1 + s_1 a_2) + s_2 b_2$$

By induction, we find sequences $a_i \in A$, $s_i \in \mathfrak{a}$, and $b_i \in B$. The sequence

$$r_n = \sum_{k \leq n} \left(\prod_{i \leq k} s_i \right) a_k$$

converges to an element $r \in A$ because A is complete. Since B is separated, the difference

$$b - \varphi(r_n) = \prod_{k \leq n+1} s_k b_{n+1}$$

tends to zero. Therefore, $b = \varphi(r)$. \square

Proposition 40. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ a principal ideal such that A is \mathfrak{a} -adically complete. Let $\phi: B \rightarrow \bar{B}$ and $\psi: B_0 \rightarrow \bar{B}_0$ be surjective A -homomorphisms of formally finitely generated A -algebras. Let $\varepsilon: A\{X\} \rightarrow B$ be a surjective A -homomorphism, where $X = \{x_1, \dots, x_n\}$ is a finite set, and $J = \ker \varepsilon$, $\bar{J} = \ker(\phi \circ \varepsilon)$. Assume that l, r, c, h, n are natural numbers such that*

1. $\mathfrak{a}^r B \cap \text{ann}_B(t) = 0$ and $\mathfrak{a}^r \bar{B} \cap \text{ann}_{\bar{B}}(t) = 0$.
2. $\mathfrak{a}^k B \cap I = \mathfrak{a}^{k-c}(\mathfrak{a}^c B \cap I)$ for all $k \geq c$.
3. $\mathfrak{a}^h \subseteq \bar{H}_{B/A}$ and $\mathfrak{a}^h \subseteq \bar{H}_{\bar{B}/A}$.
4. $\mathfrak{a}^d \{X\} \cap J = \mathfrak{a}^{d-l}(\mathfrak{a}^l \{X\} \cap J)$ and $\mathfrak{a}^d \{X\} \cap \bar{J} = \mathfrak{a}^{d-l}(\mathfrak{a}^l \{X\} \cap \bar{J})$ for all $d \geq l$.
5. $n > \max(r + 2h, c + 2h, l + 2h, 4h)$.

and there are isomorphisms $\alpha_n: B_0/\mathfrak{a}^n B_0 \rightarrow B/\mathfrak{a}^n B$ and $\beta_n: \bar{B}_0/\mathfrak{a}^n \bar{B}_0 \rightarrow \bar{B}/\mathfrak{a}^n \bar{B}$ such that the following diagram is commutative

$$\begin{array}{ccc} B_0/\mathfrak{a}^n B_0 & \xrightarrow{\psi_n} & \bar{B}_0/\mathfrak{a}^n \bar{B}_0 \\ \downarrow \alpha_n & & \downarrow \beta_n \\ B/\mathfrak{a}^n B & \xrightarrow{\phi_n} & \bar{B}/\mathfrak{a}^n \bar{B} \end{array}$$

Then, there exist isomorphisms $\alpha: B_0 \rightarrow B$ and $\beta: \bar{B}_0 \rightarrow \bar{B}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
& & B_0 & \xrightarrow{\psi} & \bar{B}_0 \\
& \swarrow \alpha & \downarrow \phi & & \searrow \beta \\
B & \xrightarrow{\quad} & \bar{B} & & \\
\downarrow & & \downarrow & & \downarrow \\
B_0/\mathfrak{a}^{n-2h} B_0 & \xrightarrow{\psi_{n-2h}} & \bar{B}_0/\mathfrak{a}^{n-2h} \bar{B}_0 & & \\
\swarrow \alpha_{n-2h} & & \searrow \beta_{n-2h} & & \\
B/\mathfrak{a}^{n-2h} B & \xrightarrow{\phi_{n-2h}} & \bar{B}/\mathfrak{a}^{n-2h} \bar{B} & &
\end{array}$$

Proof. By Theorem 26, there exist A -homomorphisms $\alpha: B_0 \rightarrow B$ and $\beta: \bar{B}_0 \rightarrow \bar{B}$ such that in the following diagram right cube is commutative

$$\begin{array}{ccccccc}
& \bar{J}_0 & \xrightarrow{\quad} & J_0 & \xrightarrow{\quad} & A\{X\} & \xrightarrow{\varepsilon_0} B_0 \xrightarrow{\psi} \bar{B}_0 \\
& \swarrow & \downarrow & \swarrow & \downarrow & \downarrow & \swarrow \alpha \quad \searrow \beta \\
\bar{J} & \xrightarrow{\quad} & J & \xrightarrow{\quad} & A\{X\} & \xrightarrow{\varepsilon} B & \xrightarrow{\phi} \bar{B} \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \downarrow & \downarrow \\
\bar{J}_0/\bar{J}_0 \cap \mathfrak{a}^m\{X\} & \xrightarrow{\quad} & J_0/J_0 \cap \mathfrak{a}^m\{X\} & \xrightarrow{\quad} & A/\mathfrak{a}^m[X] & \xrightarrow{\alpha_m} B_0/\mathfrak{a}^m B_0 \xrightarrow{\psi_m} \bar{B}_0/\mathfrak{a}^m \bar{B}_0 \\
\swarrow & \downarrow & \swarrow & \downarrow & \downarrow & \downarrow & \swarrow \beta_m \\
\bar{J}/\bar{J} \cap \mathfrak{a}^m\{X\} & \xrightarrow{\quad} & J/J \cap \mathfrak{a}^m\{X\} & \xrightarrow{\quad} & A/\mathfrak{a}^m[X] & \xrightarrow{\phi_m} B/\mathfrak{a}^m B & \xrightarrow{\quad} \bar{B}/\mathfrak{a}^m \bar{B}
\end{array}$$

where $m = n - 2h$. By Lemma 39, the map α is surjective, and we define ε_0 to be a lifting of ε . Again, by Lemma 39, the map ε_0 is surjective, and we define $J_0 = \ker \varepsilon_0$ and $\bar{J}_0 = \ker(\psi \circ \varepsilon_0)$. Therefore, we have constructed the diagram above such that it becomes commutative. By construction, $J_0 \subseteq J$ and $\bar{J}_0 \subseteq \bar{J}$. Since the left cube of the diagram is commutative, we have

$$\begin{aligned}
J &= J_0 + J \cap \mathfrak{a}^m\{X\} \subseteq J_0 + \mathfrak{a} J \\
\bar{J} &= \bar{J}_0 + \bar{J} \cap \mathfrak{a}^m\{X\} \subseteq \bar{J}_0 + \mathfrak{a} \bar{J}
\end{aligned}$$

The inclusions above hold because of item 4 from the hypothesis of the proposition. Hence $J = J_0$ and $\bar{J} = \bar{J}_0$ by Nakayama's lemma. Therefore, the maps α and β are isomorphisms. \square

5 Algebraization of pairs

5.1 Pairs of modules

Lemma 41. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, and $B \rightarrow \bar{B} \rightarrow 0$ is a surjection of A -algebras. Let M and P be B -modules and \bar{M} and \bar{P} be \bar{B} -modules such that*

1. There is an exact sequence of B -modules

$$0 \rightarrow R \rightarrow M \xrightarrow{\varphi} \bar{M} \rightarrow 0;$$

2. $\pi: P \rightarrow \bar{P}$ is a homomorphism of B -modules;

3. M is \mathfrak{a} -adically complete;

4. P and \bar{P} are finite modules over B and \bar{B} , respectively.

Suppose that we are given natural numbers r, h, c , and n such that

1. $\mathfrak{a}^r M \cap \text{ann}_M(t) = 0$ and $\mathfrak{a}^r \bar{M} \cap \text{ann}_{\bar{M}}(t) = 0$;

2. $\mathfrak{a}^h \subseteq H_P \subseteq B$ and $\mathfrak{a}^h \subseteq H_{\bar{P}} \subseteq \bar{B}$;

3. $\mathfrak{a}^k M \cap R = \mathfrak{a}^{k-c}(\mathfrak{a}^c M \cap R)$ for all $k \geq c$;

4. $n > \max(r + 2h, c + 2h, 4h)$.

Let us be given the following commutative diagram

$$\begin{array}{ccc} M/\mathfrak{a}^n M & \xrightarrow{\varphi_n} & \bar{M}/\mathfrak{a}^n \bar{M} \longrightarrow 0 \\ \alpha_n \uparrow & & \uparrow \beta_n \\ P & \xrightarrow{\pi} & \bar{P} \end{array}$$

where φ_n is induced by φ (all homomorphisms on the diagram are the homomorphisms of B -modules).

Then, there exist homomorphisms $\alpha: P \rightarrow M$ and $\beta: \bar{P} \rightarrow \bar{M}$ (homomorphisms of B -modules) such that the following diagram is commutative

$$\begin{array}{ccccc} P & \xrightarrow{\quad \alpha \quad} & M & & \\ \downarrow \alpha_n & \searrow \pi & \downarrow & \searrow \varphi & \\ & \bar{P} & \xrightarrow{\quad \beta \quad} & \bar{M} & \\ & \downarrow \beta_n & & \downarrow & \\ M/\mathfrak{a}^n M & \xrightarrow{\quad \quad} & M/\mathfrak{a}^{n-2h} M & \xrightarrow{\quad \varphi_{n-2h} \quad} & \bar{M}/\mathfrak{a}^{n-2h} \bar{M} \\ & \searrow \varphi_n & & \searrow & \\ & \bar{M}/\mathfrak{a}^n \bar{M} & \xrightarrow{\quad \quad} & \bar{M}/\mathfrak{a}^{n-2h} \bar{M} & \end{array}$$

Proof. By Lemma 24, one finds homomorphisms

$$\gamma: P \rightarrow M/\mathfrak{a}^{2n-h} M \quad \text{and} \quad \delta: \bar{P} \rightarrow \bar{M}/\mathfrak{a}^{2n-h} \bar{M}$$

such that the following diagrams are commutative

$$\begin{array}{ccc}
P & \xrightarrow{\gamma} & M/\mathfrak{a}^{2n-h} M \\
\alpha_n \downarrow & \searrow \alpha_{n-h} & \downarrow \\
M/\mathfrak{a}^n M & \longrightarrow & M/\mathfrak{a}^{n-h} M
\end{array}
\quad
\begin{array}{ccc}
\bar{P} & \xrightarrow{\delta} & \bar{M}/\mathfrak{a}^{2n-h} \bar{M} \\
\beta_n \downarrow & \searrow \beta_{n-h} & \downarrow \\
\bar{M}/\mathfrak{a}^n \bar{M} & \longrightarrow & \bar{M}/\mathfrak{a}^{n-h} \bar{M}
\end{array}$$

Where, α_{n-h} and β_{n-h} are the compositions of α_n and β_n with the corresponding quotient map, respectively.

We will draw everything on the following diagram

$$\begin{array}{ccccc}
& & \mathfrak{a}^{n-2h} M/\mathfrak{a}^{2(n-2h)} M & \longrightarrow & \mathfrak{a}^{n-2h} \bar{M}/\mathfrak{a}^{2(n-2h)} \bar{M} \\
& & \searrow i & & \searrow \\
& & M/\mathfrak{a}^{2(n-2h)} M & \xrightarrow{\varphi_{2(n-2h)}} & \bar{M}/\mathfrak{a}^{2(n-2h)} \bar{M} \\
& & \downarrow \nu & & \downarrow \bar{\nu} \\
& & \mathfrak{a}^{n-h} M/\mathfrak{a}^{2n-h} M & \longrightarrow & \mathfrak{a}^{n-h} \bar{M}/\mathfrak{a}^{2n-h} \bar{M} \\
& & \searrow j & & \searrow \\
& & M/\mathfrak{a}^{2n-h} M & \xrightarrow{\varphi_{2n-h}} & \bar{M}/\mathfrak{a}^{2n-h} \bar{M} \\
& & \downarrow \gamma & & \downarrow \delta \\
& & M/\mathfrak{a}^{n-h} M & \xrightarrow{\varphi_{n-h}} & \bar{M}/\mathfrak{a}^{n-h} \bar{M} \\
& & \uparrow \eta & & \uparrow \\
& & P & \xrightarrow{\pi} & \bar{P}
\end{array}$$

α_{n-h} (arrow from P to $M/\mathfrak{a}^{n-h} M$)
 β_{n-h} (arrow from \bar{P} to $\bar{M}/\mathfrak{a}^{n-h} \bar{M}$)
 ξ (arrow from P to $M/\mathfrak{a}^{2n-h} M$)
 γ (arrow from $M/\mathfrak{a}^{2n-h} M$ to $M/\mathfrak{a}^{n-h} M$)
 δ (arrow from $\bar{M}/\mathfrak{a}^{2n-h} \bar{M}$ to $\bar{M}/\mathfrak{a}^{n-h} \bar{M}$)

On this diagram $i, j, \nu, \bar{\nu}$ are the natural maps. We will produce all dotted arrows step by step.

Note, that the diagram is commutative only for solid arrows. By construction, the homomorphism

$$\varphi_{2n-h} \circ \gamma - \delta \circ \pi$$

factors through j , that is, equals $j \circ \xi$ for some

$$\xi: P \rightarrow \mathfrak{a}^{n-h} \bar{M}/\mathfrak{a}^{2n-h} \bar{M}.$$

Then, by Lemma 23, we find the homomorphism

$$\eta: P \rightarrow \mathfrak{a}^{n-2h} M/\mathfrak{a}^{2(n-2h)} M.$$

Thus, homomorphisms

$$\nu \circ \gamma - i \circ \eta: P \rightarrow M / \mathfrak{a}^{2(n-2h)} M$$

and

$$\bar{\nu} \circ \delta: \bar{P} \rightarrow \bar{M} / \mathfrak{a}^{2(n-2h)} \bar{M}$$

give the required lifting. Now, since M and, hence, \bar{M} are \mathfrak{a} -adically complete, the result follows by induction on n . \square

Theorem 42. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, we are given a surjection $B \rightarrow \bar{B}$ of A -algebras, Q is a \bar{B} -module, and \bar{Q} is a $\widehat{\bar{B}}$ -modules such that*

1. *The pairs $(B, \mathfrak{a} B)$ and $(\bar{B}, \mathfrak{a} \bar{B})$ are henselian;*
2. *$\phi: Q \rightarrow \bar{Q}$ is a surjective homomorphism of $\widehat{\bar{B}}$ -modules;*
3. *Q_t is \widehat{B}_t -projective;*
4. *\bar{Q}_t is $\widehat{\bar{B}}_t$ -projective;*
5. *Q is a finite \widehat{B} -module.*

Then, there exist a B -module P_0 , a \bar{B} -module \bar{P}_0 , a surjection $\phi_0: P_0 \rightarrow \bar{P}_0$, and isomorphisms $\varphi: Q \rightarrow \widehat{P}_0$ and $\psi: \bar{Q} \rightarrow \widehat{\bar{P}}_0$ such that the following diagram is commutative

$$\begin{array}{ccccc} Q & \xrightarrow{\phi} & \bar{Q} & \longrightarrow & 0 \\ \downarrow \varphi & & \downarrow \psi & & \\ \widehat{P}_0 & \xrightarrow{\widehat{\phi}_0} & \widehat{\bar{P}}_0 & \longrightarrow & 0 \end{array}$$

Proof. By Theorem 3 of [3], we can algebraize Q and \bar{Q} separately. So, we may suppose that $Q = \widehat{P}$, $\bar{Q} = \widehat{\bar{P}}$, where P is a finite B -module, \bar{P} is a finite \bar{B} -module, and, for some natural h , we have

$$\mathfrak{a}^h \subseteq H_P \subseteq B \quad \text{and} \quad \mathfrak{a}^h \subseteq H_{\bar{P}} \subseteq \bar{B}.$$

We take some free resolutions of P and \bar{P} as follows

$$B^p \xrightarrow{L} B^q \longrightarrow P \longrightarrow 0$$

$$\bar{B}^{\bar{p}} \xrightarrow{\bar{L}} \bar{B}^{\bar{q}} \longrightarrow \bar{P} \longrightarrow 0$$

Then the homomorphism ϕ induces the homomorphisms u and v on the following diagram

$$\begin{array}{ccccc}
& 0 & & 0 & \\
& \uparrow & & \uparrow & \\
\widehat{P} & \xrightarrow{\phi} & \widehat{P} & \longrightarrow & 0 \\
& \uparrow & & \uparrow & \\
\widehat{B}^q & \xrightarrow{u} & \widehat{B}^{\bar{q}} & \longrightarrow & 0 \\
& \uparrow & & \uparrow & \\
L & & \bar{L} & & \\
& \uparrow & & \uparrow & \\
\widehat{B}^p & \xrightarrow{v} & \widehat{B}^{\bar{p}} & &
\end{array}$$

where we may suppose that u is surjective by enlarging the q . Here, L is a $q \times p$ -matrix over B and \bar{L} is $\bar{q} \times \bar{p}$ -matrix over \bar{B} .

We define the following algebra

$$D = \bar{B}[X, Y]/(XL - \bar{L}Y),$$

where X is a $\bar{q} \times q$ -matrix of indeterminates x_{ij} , and Y is a $\bar{p} \times p$ -matrix of indeterminates y_{ij} . This algebra describes all pairs of B -homomorphisms u_0, v_0 such that the following diagram is commutative

$$\begin{array}{ccc}
B^q & \xrightarrow{u_0} & \bar{B}^{\bar{q}} \\
L \uparrow & & \uparrow \bar{L} \\
B^p & \xrightarrow{v_0} & \bar{B}^{\bar{p}}
\end{array}$$

The algebra $\bar{D} = \widehat{\bar{B}} \otimes_{\bar{B}} D$ describes the same pairs of \widehat{B} -homomorphisms for free modules over \widehat{B} and $\widehat{\bar{B}}$.

The pair u and v defines a \widehat{B} -section $\bar{D} \rightarrow \widehat{\bar{B}}$. So, we have

$$\begin{array}{ccc}
\widehat{\bar{B}} & \xleftarrow{(u,v)} & \bar{D} \\
\uparrow & & \uparrow \\
\bar{B} & \xleftarrow{(u_0, v_0)} & D
\end{array}$$

The algebra D_t is smooth over \bar{B}_t by Lemma 43 below. By Theorem 2 bis of [3], there is a section $D \rightarrow \bar{B}$ given by a pair (u_0, v_0) such that the following diagrams are commutative

$$\begin{array}{ccc}
& \widehat{\bar{B}}^{\bar{q}} & \\
\widehat{B}^q \nearrow \widehat{u_0} & & \searrow \\
& \left(\widehat{\bar{B}} / \mathfrak{a}^n \widehat{\bar{B}} \right)^{\bar{q}} & \\
\widehat{B}^q \searrow u & & \nearrow \widehat{\bar{B}}^{\bar{q}}
\end{array}
\quad
\begin{array}{ccc}
& \widehat{\bar{B}}^{\bar{p}} & \\
\widehat{B}^p \nearrow \widehat{v_0} & & \searrow \\
& \left(\widehat{\bar{B}} / \mathfrak{a}^n \widehat{\bar{B}} \right)^{\bar{p}} & \\
\widehat{B}^p \searrow v & & \nearrow \widehat{\bar{B}}^{\bar{p}}
\end{array}$$

for sufficiently large n . In particular,

$$\text{Im } \widehat{u}_0 + \text{Rad } \widehat{B}^{\bar{q}} = \text{Im } u = \widehat{B}^{\bar{q}}$$

By Nakayama's lemma, \widehat{u}_0 is also surjective. Since B is henselian, the completion \widehat{B} is faithfully flat. Therefore, u_0 is also surjective.

In other words, the pair (u_0, v_0) induces a surjective B -homomorphism $P \xrightarrow{\phi_0} \widehat{P} \rightarrow 0$ and, by definition, we have the following commutative diagram

$$\begin{array}{ccccc} \widehat{P} & \xrightarrow{\widehat{\phi}_0} & \widehat{\widehat{P}} & \longrightarrow & 0 \\ \downarrow \varphi_n & & \downarrow \psi_n & & \\ \widehat{P}/\mathfrak{a}^n \widehat{P} & \xrightarrow{\phi_n} & \widehat{\widehat{P}}/\mathfrak{a}^n \widehat{\widehat{P}} & \longrightarrow & 0 \end{array}$$

where ϕ_n is induced by ϕ , φ_n and ψ_n are the quotient maps.

Now, by Lemma 41, we can find homomorphisms $\varphi: \widehat{P} \rightarrow \widehat{\widehat{P}}$ and $\psi: \widehat{\widehat{P}} \rightarrow \widehat{\widehat{\widehat{P}}}$ such that the following diagram is commutative

$$\begin{array}{ccccc} \widehat{P} & \xrightarrow{\phi_0} & \widehat{\widehat{P}} & \longrightarrow & 0 \\ \downarrow \varphi & & \downarrow \psi & & \\ \widehat{\widehat{P}} & \xrightarrow{\phi} & \widehat{\widehat{\widehat{P}}} & \longrightarrow & 0 \end{array}$$

and, by construction, φ and ψ equal φ_n and ψ_n modulo \mathfrak{a}^{n-2h} , respectively. Thus they are equal to the identity maps modulo \mathfrak{a}^{n-2h} and therefore are also isomorphisms. \square

Lemma 43. *Let the conditions of Theorem 42 hold and the \bar{B} -algebra D is defined as in the proof of the proposition. Then, for every prime ideal $\mathfrak{p} \subseteq \bar{B}$ not containing t , $D_{\mathfrak{p}}$ is a polynomial ring over $\bar{B}_{\mathfrak{p}}$.*

Proof. The matrix L also defines the \bar{B} -module $P' = P \otimes_B \bar{B}$. This module is projective over the complement of $V(\mathfrak{a})$. Therefore, we have the following split exact sequence of free $\bar{B}_{\mathfrak{p}}$ -modules

$$0 \rightarrow K_{\mathfrak{p}} \rightarrow \bar{B}_{\mathfrak{p}}^q \rightarrow P'_{\mathfrak{p}} \rightarrow 0$$

where K is the module generated by the columns of L . Since $K_{\mathfrak{p}}$ is free of some rank d and $\bar{B}_{\mathfrak{p}}$ is local, one can find matrices $U \in \text{GL}_q(\bar{B}_{\mathfrak{p}})$ and $V \in \text{GL}_p(\bar{B}_{\mathfrak{p}})$ such that the matrix ULV is of the form

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

where E is the identity matrix of size d . By the same arguments, there exist matrices $\bar{U} \in \text{GL}_{\bar{q}}(\bar{B}_{\mathfrak{p}})$ and $\bar{V} \in \text{GL}_{\bar{p}}(\bar{B}_{\mathfrak{p}})$ such that the matrix $\bar{U}\bar{L}\bar{V}$ is of the form

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

where E is the identity matrix of some size \bar{d} . Now, we change the generators X and Y by $X' = \bar{U}XU^{-1}$ and $Y' = \bar{V}^{-1}YV$, respectively. Then, the algebra $D_{\mathfrak{p}}$ is given by

$$D_{\mathfrak{p}} = \bar{B}_{\mathfrak{p}}[X', Y']/(F)$$

where F is a $\bar{q} \times p$ -matrix of the form

$$\begin{pmatrix} X'_d & 0 \\ Y'_d & 0 \end{pmatrix}$$

Here X'_d is the matrix consisting of the first d columns of the matrix X' and Y'_d is the matrix consisting of the first \bar{d} rows of the matrix Y' . Now, the claim is clear. \square

Corollary 44. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, $f: B \rightarrow \bar{B}$ is a surjection of A -algebras, Q is a \hat{B} -module, and \bar{Q} is a $\hat{\bar{B}}$ -modules such that*

1. $\phi: Q \rightarrow \bar{Q}$ is a surjective homomorphism of \hat{B} -modules;
2. Q_t is \hat{B}_t -projective;
3. \bar{Q}_t is $\hat{\bar{B}}_t$ -projective;
4. Q is a finite \hat{B} -module.

Then, there exists a surjective homomorphism of A -algebras $f': B' \rightarrow \bar{B}'$ such that the following diagram is commutative

$$\begin{array}{ccccc} B' & \xrightarrow{f'} & \bar{B}' & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ B & \xrightarrow{f} & \bar{B} & \longrightarrow & 0 \end{array}$$

where B' and \bar{B}' are étale extensions of B and \bar{B} , respectively. There also exist a B' -module P_0 , a \bar{B}' -module \bar{P}_0 , a surjection $P_0 \xrightarrow{\phi_0} \bar{P}_0 \rightarrow 0$, and isomorphisms $\varphi: Q \rightarrow \widehat{P_0}$ and $\psi: \bar{Q} \rightarrow \widehat{\bar{P}_0}$ such that the following diagram is commutative

$$\begin{array}{ccccc} Q & \xrightarrow{\phi} & \bar{Q} & \longrightarrow & 0 \\ \downarrow \varphi & & \downarrow \psi & & \\ \widehat{P_0} & \xrightarrow{\widehat{\phi_0}} & \widehat{\bar{P}_0} & \longrightarrow & 0 \end{array}$$

5.2 Pairs of algebras

5.2.1 A particular case

Proposition 45. *Let A be a ring $\mathfrak{a} = (t) \subseteq A$ is a principal ideal, and we are given the following commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & \widehat{A}\{X\} & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \pi \\ 0 & \longrightarrow & \bar{J} & \longrightarrow & \widehat{A}\{X\} & \longrightarrow & \bar{B} \longrightarrow 0 \end{array}$$

where B and \bar{B} are formally finitely generated \widehat{A} -algebras, $X = \{x_1, \dots, x_n\}$, J and \bar{J} the corresponding ideals, and π is surjective. Assume that B and \bar{B} are formally smooth over the complement of $V(\widehat{\mathfrak{a}})$ and additionally we have

1. $(J/J^2)_t$ is a free B_t -module of rank d ;
2. $(\bar{J}/\bar{J}^2)_t$ is a free \bar{B}_t -module of rank \bar{d} ;
3. $(\bar{J}/\bar{J}^2)_t = J/J^2 \otimes_B \bar{B}_t \oplus D$, where D is a free \bar{B}_t -module of rank $k = \bar{d} - d$.

Then there exist a surjective homomorphism of A -algebras $\pi_0: D \rightarrow \bar{D}$, and isomorphisms $\varphi: B \rightarrow \widehat{D}$ and $\psi: \bar{B} \rightarrow \widehat{D}$ such that D and \bar{D} are smooth over the complement of $V(\mathfrak{a})$ and the following diagram is commutative

$$\begin{array}{ccccc} & B & \xrightarrow{\pi} & \bar{B} & \longrightarrow 0 \\ & \swarrow \varphi & & \searrow \psi & \\ \widehat{D} & \xrightarrow{\widehat{\pi}_0} & \widehat{D} & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ D & \xrightarrow{\pi_0} & \bar{D} & \longrightarrow & 0 \end{array}$$

Proof. Firstly, we explicitly choose some “good” generators of ideals J and \bar{J} . We take the generators

$$\begin{aligned} J &= (f_1, \dots, f_d, f_{d+1}, \dots, f_m) \\ \bar{J} &= (f_1, \dots, f_m, g_1, \dots, g_k, g_{k+1}, \dots, g_s) \end{aligned}$$

such that the images of f_1, \dots, f_d form the basis of $(J/J^2)_t$ and the images of g_1, \dots, g_k form the basis of D . So, we have

$$\begin{aligned} (J/J^2)_t &= (f_1, \dots, f_d)_t \\ (\bar{J}/\bar{J}^2)_t &= (f_1, \dots, f_d, g_1, \dots, g_k)_t \end{aligned}$$

Therefore, there exist elements $e \in J$ and $q \in \bar{J}$ and a natural h such that

$$\begin{aligned} J_t &= (f_1, \dots, f_d, e)_t \\ \bar{J}_t &= (f_1, \dots, f_d, g_1, \dots, g_k, q)_t \end{aligned}$$

and

$$\begin{aligned} e^2 &= t^h e \pmod{f_1, \dots, f_d} && \text{in } \widehat{A}\{X\} \\ q^2 &= t^h q \pmod{f_1, \dots, f_d, g_1, \dots, g_k} && \text{in } \widehat{A}\{X\} \end{aligned}$$

The condition (1) implies that there is a natural h' such that

$$t^{h'} \in \Delta^d(f_1, \dots, f_d).$$

Enlarging h or h' , we can assume that $h = h'$. The conditions (2) and (3) imply that there exists a natural h' such that

$$t^{h'} \in \Delta^{d+k}(f_1, \dots, f_d, g_1, \dots, g_k).$$

Again, we can assume that $h = h'$.

Now, we consider the following relations on f_i, g_i, e, q :

$$\begin{aligned} \Phi_1 &= e^2 - t^h e + a_1 f_1 + \dots + a_d f_d \\ \Phi_2 &= q^2 - t^h q + b_1 f_1 + \dots + b_d f_d + c_1 g_1 + \dots + c_k g_k \\ K_j &= t^h f_{d+j} + u_{j1} f_1 + \dots + u_{jd} f_d + u_j e \\ L_j &= t^h g_{k+j} + w_{j1} f_1 + \dots + w_{jd} f_d + v_{j1} g_1 + \dots + v_{jk} g_k + v_j q \end{aligned}$$

We take $A[X]^h$, that is, the henselization of $A[X]$ with respect to ideal $\mathfrak{a}[X]$, and denote by $\mathfrak{a}[X]^h$ the corresponding ideal in $A[X]^h$. We define the following algebra

$$D = A[X]^h[F_i, G_i, E, Q, A_i, B_i, C_i, U_{ij}, V_{ij}, W_{ij}, U_i, V_i]/(\Phi_1, \Phi_2, K_j, L_j)$$

where we use capital letters to denote the corresponding indeterminates. The system of equations Φ_1, Φ_2, K_j, L_j will be denoted by Σ . Also we set

$$\bar{D} = \widehat{A}\{X\} \otimes_{A[X]^h} D.$$

The elements f_i, g_i, e, q , etc. give a section $\bar{\varepsilon}: \bar{D} \rightarrow \widehat{A}\{X\}$. Let us show that

$$t^{h(m+s-d-k+2)} \in \bar{\varepsilon} \left(H_{\bar{D}/\widehat{A}\{X\}} \right)$$

For we show that

$$t^{h(m+s-d-k+2)} \in \bar{\varepsilon} \left(\Delta^{m+s-d-k+2}(\Sigma) \right)$$

Indeed, the Jacobian matrix of Σ at f_i, g_i, e, q , etc. is the following

	f_{d+j}	g_{k+j}	e	a_i	q	b_i	c_i	other indeterminates
Φ_1	0	0	$2e - t^h$	f_i	0	0	0	*
Φ_2	0	0	0	0	$2q - t^h$	f_i	g_i	*
K_1	0	0	*	0	0	0	0	*
K_j	t^h	0	*	0	0	0	0	*
K_{m-d}	0	0	*	0	0	0	0	*
L_1	0	0	0	0	*	0	0	*
L_j	0	t^h	0	0	*	0	0	*
L_{s-k}	0	0	0	0	*	0	0	*

Therefore, we have the following inclusion

$$t^{h(m-d)}(2e - t^h, f_i)t^{h(s-k)}(2q - t^h, f_i, g_j) \subseteq \bar{\varepsilon}(\Delta^{m+s-d-k+2}(\Sigma)),$$

where $1 \leq i \leq d$ and $1 \leq j \leq k$. Now, by the choice of e , we have

$$(2e - t^h)(2e - t^h) = t^{2h} \pmod{f_1, \dots, f_d} \text{ in } \hat{A}\{X\}$$

Therefore, $t^{2h} \in (2e - t^h, f_i)$. Absolutely analogous calculation shows that $t^{2h} \in (2q - t^h, f_i, g_j)$ and we get what we need.

By Theorem 2 bis of [3], the section $\bar{\varepsilon}$ can be approximated by a section $\varepsilon: D \rightarrow A[X]^h$ modulo \mathfrak{a}^n for any sufficiently large n . We denote the corresponding elements by the same letters with upper index 0, that is, f_i^0, g_i^0, e^0, q^0 , etc. and define the following ideals and algebras

$$\begin{aligned} J^0 &= (f_1^0, \dots, f_m^0, e^0) \subseteq A[x]^h & \bar{J}^0 &= (f_1^0, \dots, f_m^0, e^0, g_1^0, \dots, g_s^0, q^0) \subseteq A[x]^h \\ B^0 &= A[X]^h / J^0 & \bar{B}^0 &= A[X]^h / \bar{J}^0 \end{aligned}$$

Because of the choice of f_i^0 and e^0 , we have

$$\Delta^d(f_1^0, \dots, f_d^0) = \Delta^d(f_1, \dots, f_d) \pmod{\hat{\mathfrak{a}}^n\{X\}}$$

If $n > h$, then t^h belongs to the left-hand part. Similarly, we get that t^h belongs to $\Delta^{d+k}(f_1^0, \dots, f_d^0, g_1^0, \dots, g_k^0)$. Because of K_j and L_j , we see that

$$J_t^0 = (f_1^0, \dots, f_d^0, e^0)_t \text{ and } \bar{J}_t^0 = (f_1^0, \dots, f_d^0, g_1^0, \dots, g_k^0, q^0)_t$$

Since e^0/t^h is idempotent modulo (f_i^0) and q^0/t^h is idempotent modulo (f_i^0, g_j^0) , the algebras B^0 and \bar{B}^0 are smooth over the complement of $V(\mathfrak{a})$ by the Jacobian criterion [5, Theorem 22.6.1].

By definition, we have a natural surjection $\pi^0: B^0 \rightarrow \bar{B}^0$, and the following diagram is commutative

$$\begin{array}{ccccc} B^0 & \xrightarrow{\pi^0} & \bar{B}^0 & \longrightarrow & 0 \\ \downarrow \alpha_n & & \downarrow \bar{\alpha}_n & & \\ B^0 / \mathfrak{a}^n & \xrightarrow{\pi_n^0} & \bar{B}^0 / \mathfrak{a}^n & \longrightarrow & 0 \\ \parallel & & \parallel & & \\ B / \mathfrak{a}^n & \xrightarrow{\pi_n} & \bar{B} / \mathfrak{a}^n & \longrightarrow & 0 \end{array}$$

where π_n is induced by π , π_n^0 is induced by π^0 , and the vertical maps α_n and $\bar{\alpha}_n$ are the quotient maps. Then, by Proposition 40, there exists isomorphisms

$\alpha: B^0 \rightarrow B$ and $\bar{\alpha}: \bar{B}^0 \rightarrow \bar{B}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
& \widehat{B^0} & \xrightarrow{\psi} & \widehat{\bar{B}^0} & \\
& \swarrow \alpha & & \nwarrow \beta & \\
B & \xrightarrow{\phi} & \bar{B} & & \\
\downarrow & & \downarrow & & \downarrow \\
& B_0/\mathfrak{a}^{n-2h} B_0 & \xrightarrow{\psi_{n-2h}} & \bar{B}_0/\mathfrak{a}^{n-2h} \bar{B}_0 & \\
\swarrow \alpha_{n-2h} & & \searrow \beta_{n-2h} & & \\
B/\mathfrak{a}^{n-2h} B & \xrightarrow{\phi_{n-2h}} & \bar{B}/\mathfrak{a}^{n-2h} \bar{B} & &
\end{array}$$

Algebras B^0 and \bar{B}^0 are quotients of $A[X]^h$ by some finitely generated ideals. Also, expressing t^h as an element of the ideals

$$\Delta^d(f_1^0, \dots, f_d^0) \quad \text{and} \quad \Delta^{d+k}(f_1^0, \dots, f_d^0, g_1^0, \dots, g_k^0)$$

we use finitely many elements of $A[x]^h$. So, we can take a strictly étale extension $A[X]^0$ of $A[X]$ containing all these elements and solutions of Σ in $A[x]^h$. Then the required algebras are

$$D = A[X]^0 / (J^0 \cap A[X]^0) \quad \text{and} \quad \bar{D} = A[X]^0 / (\bar{J}^0 \cap A[X]^0)$$

and the map $\pi_0: D \rightarrow \bar{D}$ is the natural quotient map. □

5.2.2 Reduction of the general case

If we are given a ring A with an ideal \mathfrak{a} , B is an \mathfrak{a} -adically complete A -algebra, and M is a B -module. Then the symmetric algebra of M over B will be denoted by $S_B(M)$. The completion of $S_B(M)$ in the topology generated by \mathfrak{a} will be denoted by $S_B\{M\}$.

Lemma 46. *Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal, and we are given two A -algebras B and C together with two homomorphisms $\Phi, \Psi: B \rightarrow C$ over A such that*

1. B is $\mathfrak{a}B$ -adically complete;
2. C is $\mathfrak{a}C$ -adically complete;
3. Ψ is an isomorphism;
4. $\text{Im } \gamma \subseteq \mathfrak{a}C$, where $\gamma = \Phi - \Psi$.

Then Φ is also an isomorphism.

Proof. To prove the result, we should show that $\Phi = \Psi + \gamma$ has an inverse map. We define

$$\varphi = \Psi^{-1} \circ \gamma: B \rightarrow B$$

and we have $\Phi = \Psi \circ (Id_B + \varphi)$. It is enough to show that $Id_B + \varphi$ is an invertible homomorphism of A -modules. Since Ψ^{-1} is also an A -homomorphism, $\text{Im } \varphi \subseteq \mathfrak{a}B$. The ring B is $\mathfrak{a}B$ -adically complete and, for any element $x \in B$, $\varphi(x) \in \mathfrak{a}B$. Therefore, the element

$$\xi(x) = \sum_{n=0}^{\infty} (-1)^n \varphi^n(x)$$

is well-defined. Since Φ , Ψ , and Ψ^{-1} are A -homomorphisms, they are continuous. Therefore, the map φ is also continuous. Now, an easy calculation shows that $\xi = (Id_B + \varphi)^{-1}$. \square

Lemma 47. *Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal, and B is a finitely generated A -algebra such that its completion \widehat{B} is smooth over the complement of $V(\widehat{\mathfrak{a}})$. Then there exists an $s \in \mathfrak{a}$ such that B_{1+s} is already smooth over the complement of $V(\mathfrak{a})$.*

Proof. We represent algebra B as the following quotient $A[x]/J$, where $x = \{x_1, \dots, x_n\}$ is a finite set of indeterminates. Let us denote the localization $B_{1+\mathfrak{a}B}$ by B' and $J_{1+\mathfrak{a}B}$ by J' . Then we have the following sequence of homomorphisms $B \rightarrow B' \rightarrow \widehat{B}$, and the following exact sequence

$$0 \rightarrow \widehat{J}' \rightarrow \widehat{A}\{x\} \rightarrow \widehat{B} \rightarrow 0$$

Thus, by Lemmas 3 and 9, we have

$$\begin{aligned} H_{B'/A} &= \text{ann}_B \left(\text{Hom}_{B'} (J'/J'^2, J'/J'^2) / \text{Hom}_{A[x]} (\Omega_{A[x]/A}, J'/J'^2) \right) \\ \bar{H}_{\widehat{B}/\widehat{A}} &= \text{ann}_{\widehat{B}} \left(\text{Hom}_{\widehat{B}} (\widehat{J}'/\widehat{J}'^2, \widehat{J}'/\widehat{J}'^2) / \text{Hom}_{\widehat{A}\{x\}} (\Omega_{\widehat{A}\{x\}/\widehat{A}}^s, \widehat{J}'/\widehat{J}'^2) \right) \end{aligned}$$

Since $\widehat{B} = \widehat{B}'$ is a faithfully flat B' -module and the modules J'/J'^2 and $\Omega_{A[x]/A}$ are finitely generated, we have

$$\bar{H}_{\widehat{B}/\widehat{A}} = H_{B'/A} \otimes_{B'} \widehat{B} = H_{B'/A} \widehat{B}$$

and

$$B' \cap \bar{H}_{\widehat{B}/\widehat{A}} = H_{B'/A}.$$

Formal smoothness of \widehat{B} over the complement of $V(\widehat{\mathfrak{a}})$ means that, for some h , $\mathfrak{a}^h \subseteq \bar{H}_{\widehat{B}/\widehat{A}}$. Therefore, $\mathfrak{a}^h \subseteq H_{B'/A}$. The ideal \mathfrak{a}^h is finitely generated, hence, there is an element $s \in \mathfrak{a}$ such that $\mathfrak{a}^h \subseteq H_{B_{1+s}/A}$. \square

Lemma 48. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, and we are given a surjective homomorphism $B \rightarrow \bar{B}$ of \mathfrak{a} -adically complete \widehat{A} -algebras. Let P and \bar{P} be finite modules over B and \bar{B} , respectively, such that*

1. There is a surjective homomorphism of B -modules $P \rightarrow \bar{P}$;
2. P_t is B_t -projective;
3. \bar{P}_t is \bar{B}_t -projective.

Suppose that there exist a surjective homomorphism of finitely generated A -algebras $C \rightarrow \bar{C}$ together with isomorphisms $\varphi': S_B\{P\} \rightarrow \hat{C}$ and $\psi': S_{\bar{B}}\{\bar{P}\} \rightarrow \hat{\bar{C}}$ such that C_t and \bar{C}_t are A_t -smooth and the following diagram is commutative

$$\begin{array}{ccccc}
 S_B\{P\} & \xrightarrow{\quad} & S_{\bar{B}}\{\bar{P}\} & \longrightarrow & 0 \\
 \swarrow \varphi' & & \swarrow \psi' & & \\
 \hat{C} & \xrightarrow{\quad} & \hat{\bar{C}} & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \\
 C & \xrightarrow{\quad} & \bar{C} & \longrightarrow & 0
 \end{array}$$

Then there exist a surjective homomorphism of finitely generated A -algebras $D \rightarrow \bar{D}$ together with isomorphisms $\varphi: B \rightarrow \hat{D}$ and $\psi: \bar{B} \rightarrow \hat{\bar{D}}$ such that D_t and \bar{D}_t are A_t -smooth and the following diagram is commutative

$$\begin{array}{ccccc}
 B & \xrightarrow{\quad} & \bar{B} & \longrightarrow & 0 \\
 \swarrow \varphi & & \swarrow \psi & & \\
 \hat{D} & \xrightarrow{\quad} & \hat{\bar{D}} & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \\
 D & \xrightarrow{\quad} & \bar{D} & \longrightarrow & 0
 \end{array}$$

If additionally, B and \bar{B} are formally smooth over the complement of $V(\hat{\mathfrak{a}})$, the algebras D_t and \bar{D}_t are A_t -smooth.

Proof. Let us note that, by Lemma 47, it is enough to show the existence of such algebras D and \bar{D} without showing the smoothness condition. Now, we denote $S_B\{P\}$ and $S_{\bar{B}}\{\bar{P}\}$ by R and \bar{R} , respectively, $R \otimes_B P$ and $\bar{R} \otimes_{\bar{B}} \bar{P}$ by M and \bar{M} , respectively.

From Corollary 44, it follows that we can replace C and \bar{C} by some étale extensions such that, for some finite C -module M_0 , some finite \bar{C} -module \bar{M}_0 , and a surjective homomorphism $\phi_0: M_0 \rightarrow \bar{M}_0$, there are isomorphisms $\varphi: M \rightarrow \hat{M}_0$ and $\psi: \bar{M} \rightarrow \hat{\bar{M}}_0$ such that the following diagram is commutative

$$\begin{array}{ccccc}
 M & \xrightarrow{\quad} & \bar{M} & \longrightarrow & 0 \\
 \downarrow \varphi & & \downarrow \psi & & \\
 \hat{M}_0 & \xrightarrow{\hat{\phi}_0} & \hat{\bar{M}}_0 & \longrightarrow & 0
 \end{array}$$

The homomorphism ϕ_0 induces the following surjective homomorphism of rings

$$S_C(M_0) \rightarrow S_{\bar{C}}(\bar{M}_0) \rightarrow 0$$

We can write down the following sequence of isomorphisms for their completions

$$\begin{array}{ccccc}
\widehat{S_C(M_0)} & \longrightarrow & \widehat{S_{\bar{C}}(\bar{M}_0)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
S_{\widehat{C}}\{\widehat{M_0}\} & \longrightarrow & S_{\widehat{\bar{C}}}\{\widehat{\bar{M}_0}\} & \longrightarrow & 0 \\
\downarrow S_{\widehat{C}}\{\varphi\} & & \downarrow S_{\widehat{\bar{C}}}\{\psi\} & & \downarrow \\
S_R\{M\} & \longrightarrow & S_{\bar{R}}\{\bar{M}\} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{R \otimes_B R} & \longrightarrow & \widehat{\bar{R} \otimes_{\bar{B}} \bar{R}} & \longrightarrow & 0
\end{array}$$

ν (left curved arrow from $\widehat{S_C(M_0)}$ to $\widehat{R \otimes_B R}$)
 $\bar{\nu}$ (right curved arrow from $\widehat{S_{\bar{C}}(\bar{M}_0)}$ to $\widehat{\bar{R} \otimes_{\bar{B}} \bar{R}}$)

where the vertical arrows are isomorphisms over \widehat{A} . Then the product maps

$$\text{pr}: R \otimes_B R \rightarrow B \quad \text{and} \quad \text{pr}: \bar{R} \otimes_{\bar{B}} \bar{R} \rightarrow \bar{B}$$

induce the corresponding homomorphisms on the completions and, thus, give the following commutative diagram

$$\begin{array}{ccc}
\widehat{R \otimes_B R} & \longrightarrow & \widehat{\bar{R} \otimes_{\bar{B}} \bar{R}} \longrightarrow 0 \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
R & \longrightarrow & \bar{R} \longrightarrow 0
\end{array}$$

So, we have the following commutative cube

$$\begin{array}{ccccc}
& S_R\{\widehat{M_0}\} & \longrightarrow & S_{\bar{R}}\{\widehat{\bar{M}_0}\} & \longrightarrow 0 \\
& \uparrow \Psi & & \uparrow \bar{\Psi} & \\
R & \longrightarrow & \bar{R} & \longrightarrow & 0 \\
& \downarrow S_C(M_0) & & \downarrow S_{\bar{C}}(\bar{M}_0) & \\
C & \longrightarrow & \bar{C} & \longrightarrow & 0
\end{array}$$

where $\Psi = \text{pr} \circ \nu$ and $\bar{\Psi} = \text{pr} \circ \bar{\nu}$. By Corollary 29, for any natural number n , we can replace C and \bar{C} by some étale extensions such that there exist

homomorphisms $\Phi: S_C(M_0) \rightarrow C$ and $\bar{\Phi}: S_{\bar{C}}(\bar{M}_0) \rightarrow \bar{C}$ such that Φ and $\bar{\Phi}$ coincide with Ψ and $\bar{\Psi}$ modulo \mathfrak{a}^n and the square

$$\begin{array}{ccccc} S_C(M_0) & \longrightarrow & S_{\bar{C}}(\bar{M}_0) & \longrightarrow & 0 \\ \downarrow \Phi & & \downarrow \bar{\Phi} & & \\ C & \longrightarrow & \bar{C} & \longrightarrow & 0 \end{array}$$

is commutative. For our purpose, it is enough to take $n = 1$. By construction, the restrictions of Ψ and $\bar{\Psi}$ to $S_B\{\widehat{M}_0\}$ and $S_{\bar{B}}\{\widehat{M}_0\}$, respectively, induce isomorphisms onto R and \bar{R} , respectively. Since Φ and $\bar{\Phi}$ coincide with Ψ and $\bar{\Psi}$ modulo \mathfrak{a}^n , by Lemma 46, the restrictions of $\hat{\Phi}$ and $\hat{\bar{\Phi}}$ to the same subrings are also isomorphisms. Denoting this restrictions by the same names, we get the following commutative diagram

$$\begin{array}{ccccccc} & & S_B\{\widehat{M}_0\} & \longrightarrow & S_{\bar{B}}\{\widehat{M}_0\} & \longrightarrow & 0 \\ & \nearrow \hat{\Phi} & \downarrow \pi & & \downarrow \bar{\pi} & \nearrow \hat{\bar{\Phi}} & \\ R & \longrightarrow & \bar{R} & \longrightarrow & 0 & & \\ \downarrow p & & \downarrow \bar{p} & & \downarrow & & \\ & \nearrow \Phi' & B & \longrightarrow & \bar{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ C/\Phi(M_0) & \longrightarrow & \bar{C}/\bar{\Phi}(\bar{M}_0) & \longrightarrow & 0 & & \end{array}$$

where π and $\bar{\pi}$ are the quotient maps by the ideals generated by \widehat{M}_0 and $\widehat{\bar{M}}_0$, respectively, p and \bar{p} are the quotient maps by the ideals generated by $\hat{\Phi}(\widehat{M}_0)$ and $\hat{\bar{\Phi}}(\widehat{\bar{M}}_0)$, respectively, and Φ' and $\bar{\Phi}'$ are the isomorphisms induced by $\hat{\Phi}$ and $\hat{\bar{\Phi}}$, respectively. We set

$$D = C/\Phi(M_0) \quad \text{and} \quad \bar{D} = \bar{C}/\bar{\Phi}(\bar{M}_0).$$

Now, if B and \bar{B} are formally smooth over the complement of $V(\widehat{\mathfrak{a}})$, then by Lemma 47, we can suppose that their algebraizations D and \bar{D} are also smooth over the complement of $V(\mathfrak{a})$. □

Theorem 49. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal idea, and we are given a surjective homomorphism $B \rightarrow \bar{B}$ of formally finitely generated \widehat{A} -algebras such that B and \bar{B} are formally smooth over the complement of $V(\widehat{\mathfrak{a}})$. Then there exist a surjective homomorphism $D \rightarrow \bar{D}$ of finitely generated A -algebras being smooth over the complement of $V(\mathfrak{a})$ and two isomorphisms $\varphi: B \rightarrow \widehat{D}$ and*

$\psi: \bar{B} \rightarrow \widehat{\bar{D}}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
& B & \longrightarrow & \bar{B} & \longrightarrow 0 \\
& \searrow \varphi & & \searrow \psi & \\
\widehat{D} & \longrightarrow & \widehat{\bar{D}} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
D & \longrightarrow & \bar{D} & \longrightarrow & 0
\end{array}$$

Proof. Using Lemma 48, we will reduce this theorem to Proposition 45.

Firstly, we present algebras B and \bar{B} as quotients of a ring of the form $\widehat{A}\{x\}$, where $x = \{x_1, \dots, x_n\}$ is a finite set of indeterminates, as follows

$$\begin{array}{ccccccc}
0 & \rightarrow & J & \rightarrow & \widehat{A}\{x\} & \rightarrow & B \rightarrow 0 \\
& & \downarrow & & \parallel & & \downarrow \\
0 & \rightarrow & \bar{J} & \rightarrow & \widehat{A}\{x\} & \rightarrow & \bar{B} \rightarrow 0
\end{array}$$

Considering \bar{J}/\bar{J}^2 as a B module, we find an epimorphism $B^m \rightarrow \bar{J}/\bar{J}^2$. By Lemma 48, we can replace the homomorphism $B \rightarrow \bar{B} \rightarrow 0$ by the homomorphism $S_B\{B^m\} \rightarrow S_{\bar{B}}\{\bar{J}/\bar{J}^2\} \rightarrow 0$. So, we may assume that $\Omega_{\bar{B}/\widehat{A}}^s$ is free over the complement of $V(\widehat{\mathfrak{a}})$ of some rank \bar{d} .

Again by Lemma 48, we can replace $B \rightarrow \bar{B} \rightarrow 0$ by the composition $S_B\{J/J^2\} \rightarrow B \rightarrow \bar{B}$ and assume that $\Omega_{B/\widehat{A}}^s$ is free over the complement of $V(\widehat{\mathfrak{a}})$ of some rank d .

If we present our algebras in the form

$$B = B\{t_1, \dots, t_s\}/(t_1, \dots, t_s) \quad \text{and} \quad \bar{B} = \bar{B}\{t_1, \dots, t_s\}/(t_1, \dots, t_s),$$

where $s \geq \max(d, \bar{d})$. Then, repeating the proof of Lemma 3 of [3] with Ω^s instead of Ω , we can assume that the modules $(J/J^2)_t$ and $(\bar{J}/\bar{J}^2)_t$ are free over B_t and \bar{B}_t , respectively.

Now, we write down the second fundamental sequence for both algebras with respect to the chosen representations and restrict it to the complement of $V(\widehat{\mathfrak{a}})$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & (J/J^2)_t & \longrightarrow & \Omega_{\widehat{A}\{z\}/\widehat{A}}^s \otimes_{\widehat{A}\{z\}} B_t & \longrightarrow & (\Omega_{B/\widehat{A}}^s)_t \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (J/J^2) \otimes_B \bar{B}_t & \longrightarrow & \Omega_{\widehat{A}\{z\}/\widehat{A}}^s \otimes_{\widehat{A}\{z\}} \bar{B}_t & \longrightarrow & \Omega_{\bar{B}/\widehat{A}}^s \otimes_B \bar{B}_t \longrightarrow 0 \\
& & \downarrow & & \parallel & & \downarrow \\
0 & \longrightarrow & (\bar{J}/\bar{J}^2)_t & \longrightarrow & \Omega_{\widehat{A}\{z\}/\widehat{A}}^s \otimes_{\widehat{A}\{z\}} \bar{B}_t & \longrightarrow & (\Omega_{\bar{B}/\widehat{A}}^s)_t \longrightarrow 0
\end{array}$$

Here $z = \{z_1, \dots, z_n\}$ is the set of indeterminates such that J/J^2 and \bar{J}/\bar{J}^2 are free over the complement of $V(\hat{\mathfrak{a}})$. All these sequences are split exact. In particular,

$$(\bar{J}/\bar{J}^2)_t = (J/J^2) \otimes_B \bar{B}_t \oplus D$$

If we replace $B \rightarrow \bar{B} \rightarrow 0$ by the composition $B\{t_1, \dots, t_s\} \rightarrow B \rightarrow \bar{B}$, where s is greater than the rank of $(\Omega_{\bar{B}/\hat{A}}^s)_t$, we may assume that D is free over the complement of $V(\hat{\mathfrak{a}})$. This case is done by Proposition 45. \square

5.3 Algebraization of a divisor

Lemma 50. *Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal, B a finitely generated A -algebra, and I is an ideal of B . Assume that the ideal \hat{I} is invertible in \hat{B} , where the hat means the \mathfrak{a} -adic completion. Then there exists an element $s \in \mathfrak{a}$ such that the ideal I_{1+s} is invertible in B_{1+s} .*

Proof. Let us denote the algebra $B_{1+\mathfrak{a}B}$ by B' and $I_{1+\mathfrak{a}B}$ by I' . Then we have the following sequence of homomorphisms $B \rightarrow B' \rightarrow \hat{B}$. By Theorem 4 of [2, Chapter II, Section 6], the ideal I is invertible if and only if I is projective and contains a non-zero divisor.

1) We will prove that I' is invertible. Firstly, we will show that I' contains a non-zero divisor. Since $\hat{I} = I'\hat{B}$, it is enough to show that, if I' consists of zero divisors only, then $I'\hat{B}$ also consists of zero divisors. In this case, I' belongs to some associated prime $\mathfrak{p} \subseteq B'$ and $\mathfrak{p} = \text{ann}_{B'}(x)$ for some $x \in B'$. Since \hat{B} is flat, we have

$$\mathfrak{p}\hat{B} = \text{ann}_{B'}(x) \otimes_{B'} \hat{B} = \text{ann}_{\hat{B}}(x).$$

In particular, $\mathfrak{p}\hat{B}$ consists of zero divisors.

Secondly, we will show that I' is projective. Indeed, the ideal \hat{I} is projective, thus, flat. Since $\mathfrak{a}B'$ belongs to the radical of B' , \hat{B} is faithfully flat. But

$$\hat{I} = I'\hat{B} = I' \otimes_{B'} \hat{B}.$$

Hence, I' is flat. Since I' is finitely generated and B' is Noetherian, I' is projective.

2) Now, we will show that there is some $s \in \mathfrak{a}B$ such that I_{1+s} is invertible in B_{1+s} . Firstly, we will show that I_{1+s} contains a non-zero divisor for some s . If $a' \in I'$ is a non-zero divisor in B' , then, there are an $s_1 \in \mathfrak{a}B$ and $a \in I$ such that $(1 + s_1)a' = a$. The annihilator of a in B is finitely generated and is equal to zero in B' . Therefore, there is an element $s_2 \in \mathfrak{a}B$ such that a is not a zero divisor in B_{1+s_2} . Thus element $a \in I_{(1+s_1)(1+s_2)}$ is the required non-zero divisor.

Secondly, we will show that, for some $s \in \mathfrak{a}B$, the ideal I_{1+s} is projective B_{1+s} -module. By Lemma 17, it is enough to show that $(B/H_I)_{1+s} = 0$ for some $s \in \mathfrak{a}B$. We can represent I as follows

$$0 \rightarrow K \rightarrow F \rightarrow I \rightarrow 0$$

where F is a free finitely generated A -module. Then by Corollary 15, we have

$$H_I = \text{ann}_B(\text{Ext}_B^1(I, K))$$

But I' is projective B' -module, hence $H_{I'} = B'$. Since our modules are finitely generated over Noetherian ring, localization commutes with annihilators and Ext. Thus, $(B/H_I)_{1+\mathfrak{a}}B = 0$. Hence, there is an element $s_3 \in \mathfrak{a}B$ such that $(B/H_I)_{1+s_3} = 0$. Now, we should localize by the product

$$(1 + s_1)(1 + s_2)(1 + s_3).$$

□

Theorem 51. *Let A be a ring, $\mathfrak{a} = (t) \subseteq A$ a principal ideal, \bar{B} is a formally finitely generated \hat{A} -algebra being formally smooth over the complement of $V(\hat{\mathfrak{a}})$. Assume that we are given an invertible ideal $\bar{I} \subseteq \bar{B}$ such that the quotient \bar{B}/\bar{I} is formally smooth over the complement of $V(\hat{\mathfrak{a}})$. Then there exist a finitely generated A -algebra B and an invertible ideal $I \subseteq B$ such that $\hat{B} = \bar{B}$ and $I\hat{B} = \bar{I}$. Moreover, algebras B and B/I are smooth over the complement of $V(\mathfrak{a})$.*

Proof. Applying Theorem 49 to the homomorphism $\bar{B} \rightarrow \bar{B}/\bar{I}$, we get the following commutative diagram

$$\begin{array}{ccccc} & \bar{B} & \xrightarrow{\pi} & \bar{B}/\bar{I} & \longrightarrow 0 \\ & \swarrow \varphi & & \searrow \psi & \\ \hat{D} & \xrightarrow{\hat{\phi}} & \hat{D}' & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ D & \xrightarrow{\phi} & D' & \longrightarrow & 0 \end{array}$$

where π is the quotient map, D and D' are finitely generated A -algebras being smooth over the complement of $V(\mathfrak{a})$, and the maps φ and ψ are \hat{A} -isomorphisms. Now, we define $J = \ker \phi$. By Lemma 50, there is an element $s \in \mathfrak{a}D$ such that J_{1+s} is invertible in D_{1+s} . Now, the algebra $B = D_{1+s}$ and the ideal $I = J_{1+s}$ satisfy the required conditions. □

References

- [1] M. Artin. Algebraic approximation of structures over complete local rings. *Publ. Math. IHES*, 36:23–58, 1969.
- [2] N. Bourbaki. *Algèbre Commutative*. Hermann, Paris, 1961.
- [3] R. Elkik. Solutions d'équations à coefficients dans un anneau hensélien. *Annales scientifiques de L'É.N.S.*, Ser 4, 6(4):553–603, 1973.

- [4] O. Gabber and L. Ramero. *Almost Ring Theory*. Lecture notes in mathematics. Springer-Verlag, Berlin, 2003.
- [5] A. Grothendieck. Éléments de Géométrie Algébrique. *Publ. Math. IHES*, 20:5–259, 1964.
- [6] A. Grothendieck. Éléments de Géométrie Algébrique. *Publ. Math. IHES*, 32:5–361, 1967.
- [7] H. Matsumura. *Commutative algebra*. Mathematics Lecture Note Series. The Benjamin/Cummings Publishing Company, London, Amsterdam, Don Mills, Ontario, Sydney, Tokyo, 1980.
- [8] M. Temkin. Desingularization of quasi-excellent schemes in characteristic zero. *Adv. Math.*, 219:488–522.
- [9] M. Temkin. Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. *Duke Math. J.*, 161:2207–2254, 2012.